Applications of Conformal Mappings to Fluid Flow and Temperature Distribution

Fluid Flow. Denote by $\vec{v} = (p, q)$ the velocity of a steady 2-dimensional fluid flow in a domain Ω. ("Steady" means time-independent.) The flow is said to be *irrotational* if $\oint_C \left(\vec{v} \cdot \vec{t}\right) ds = 0$ for any curve C in Ω whose enclosure is completely in Ω, where $\vec{t}$ is the unit tangent vector of C. Since $\vec{t} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$, it follows that irrotationality is equivalent to $\oint_C pdx + qdy = 0$ for all C, which by Stokes's theorem is equivalent to

$$\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = 0$$

for any subdomain D in Ω. This means that $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ identically on Ω. When Ω is simply connected (*i.e.,* any loop in Ω can be continuously shrunk to a point inside Ω), we can find a function φ (called the *velocity potential*) such that $\vec{v} = (p, q) = \operatorname{grad} \varphi$. Let $\rho(x, y, t)$ be the fluid density at the point (x, y) and at time t. The law of conservation of mass yields the following *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\ (\rho \vec{v}) = 0,$$

which is an easy consequence of the divergence theorem. If the fluid is incompressible with constant density, then the continuity equation yields $\operatorname{div} \vec{v} = 0$, which means that $\Delta \varphi = 0$. Thus the velocity potential φ is harmonic. We will confine our discussion only to irrotational, incompressible, steady 2-dimensional flows with constant density. We are interested in determining the velocity $\vec{v}(x, y)$ at the point (x, y) and also the equations of the stream-lines. We will seek the harmonic function φ as the real part of a holomorphic function $F(z)$ so that $F = \varphi + i\psi$ with real part φ and imaginary part ψ. We call the function F the *complex velocity potential.* We can express the velocity $\vec{v} = \operatorname{grad} \varphi$ in terms of F as follows.

$$\vec{v} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right)$$

which by the Cauchy-Riemann equations can be rewritten as

$$\vec{v} = \left(\frac{\partial \varphi}{\partial x}, -\frac{\partial \psi}{\partial x}\right).$$

When we use the notation of complex numbers to express a vector, we get

$$\vec{v} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \psi}{\partial x} = \overline{\frac{\partial F}{\partial x}} = \overline{F'}.$$

We are going to express the streamlines as the level curves of some function $g =$ constant. Then the gradient of g is normal to the streamlines and must be perpendicular to the velocity vector $\vec{v} = \operatorname{grad} \varphi$ According to the Cauchy-Riemann equations, the gradient of ψ is perpendicular to the gradient of φ. Hence we can choose g to be ψ and the streamlines are the level curves of the imaginary part of the complex velocity potential F.

For the flow with constant speed $A > 0$ from the left to the right on the upper half-plane the complex velocity potential is $F(z) = Az$, because $\overline{F'} = A$. We now look at a more complicated example for which we will use the function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ which relates the exponential function to the sine and cosine function.

Example. Consider the fluid flow on $\mathbb{C} - [-1, 1]$ which at ∞ is a flow with speed A and angle α measured from the real-axis. Find the complex velocity potential on $\mathbb{C} - [-1, 1]$.

Solution. First use a new complex variabale z_1 and the map $z = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$ which maps $z \in \mathbb{C} - [-1, 1]$ one-one onto the set of all z_1 in $\mathbb{C}$ minus the closed unit disk. The flow in the z_1-space at ∞ is a flow with speed $2A$ and angle α measured from the real-axis, because the behavior of $\frac{z_1}{2}$ is the same as the behavior of z at ∞. We now introduce another complex variable z_2 which is related to z_1 by $z_2 = z_1 e^{-i\alpha}$. Clearly the set of all z_1 in $\mathbb{C}$ minus the closed unit disk is in one-one correspondence with the set of all z_2 in $\mathbb{C}$ minus the closed unit disk. The flow in the z_1-space at ∞ is a flow with speed $2A$ and from left to right, because

$$\overline{\frac{dF}{dz_2}} = \overline{\frac{dF}{dz_1}\frac{dz_1}{dz_2}} = \overline{\frac{dF}{dz_1}e^{i\alpha}} = \overline{\frac{dF}{dz_1}}\, e^{-i\alpha}.$$

We now introduce yet another complex variable w which is related to z_2 by $w = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$. The set of all z_2 in $\mathbb{C}$ minus the closed unit disk is in one-one correspondence with $w \in \mathbb{C} - [-1, 1]$. The flow in the w-space at ∞ is a flow with speed A and from left to right, because the behavior of $\frac{z_2}{2}$ is the same as the behavior of w at ∞. Thus the complex potential function

F in terms of w is simply Aw. We now express F in terms of our original complex variable z. We have to invert $z = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$. To solve for z_1 in terms of z, we solve the quadratic equation

$$(z_1)^2 - 2zz_1 + 1 = 0.$$

According to the formula for the two roots of a quadratic equation, the roots are

$$\frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1}.$$

Since for our one-one correspondence between z and z_1, a large positive z corresponds to a large positive z_1, we should take the plus sign and get

$$z_1 = z + \sqrt{z^2 - 1}.$$

The other root $z - \sqrt{z^2 - 1}$ must be $\frac{1}{z}$, because the transformation $z \mapsto \frac{1}{z}$ leaves z_1 unchanged. Now we get as our final answer for the complex velocity potential $F(z)$ the expression

$$F = Aw = \frac{A}{2}\left(z_2 + \frac{1}{z_2}\right) = \frac{A}{2}\left(z_1 e^{-i\alpha} + \frac{e^{i\alpha}}{z_1}\right)$$
$$= \frac{A}{2}\left(e^{-i\alpha}\left(z + \sqrt{z^2 - 1}\right) + e^{i\alpha}\left(z - \sqrt{z^2 - 1}\right)\right).$$

Temperature Distribution. We will consider only steady temperature distribution T in a domain with neither source nor sink. For boundary condition we will consider the case of two constant values with insulation along a continuous piece of the boundary. Then T is a harmonic function and its normal derivative is zero at a point in the insulated part of the boundary. The technique is to use a conformal mapping to map the domain to a vertical half-strip so that the insulated portion of the boundary is mapped to the horizontal part of the vertical half-strip and to use the real part of the target variable as the temperature distribution after normalization to fit the given two constant values.

Example. Find the temperature distribution T on the upper half-plane so that $T = 0$ on $(-\infty, -1]$ and $T = 1$ on $[1, \infty)$ and the interval $[-1, 1]$ on the boundary is insulated.

Solution. We consider the map the function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ which relates the exponential function to the sine and cosine function. It maps the open upper unit half-disk to the lower half-plane so that

(i) the upper half-circle goes to $[-1, 1]$,

(ii) the interval $(0, 1]$ goes to $[1, \infty)$,

(ii) the interval $[-1, 0)$ goes to $(-\infty, -1]$.

The exponent map $z \mapsto e^z$ maps the horizontal left half-strip $\{x < 0, 0 < y < \pi\}$ to the open upper unit half-disk so that

(i) the vertical line-segment $\{x = 0, 0 \leq y \leq \pi\}$ goes to the upper half-circle, $[-1, 1]$,

(ii) the horizontal line-segment $\{-\infty < x \leq 0, y = 0\}$ goes to the interval $[0, 1]$,

(iii) the horizontal line-segment $\{-\infty < x \leq 0, y = \pi\}$ goes to the interval $[-1, 0]$.

Multiplication by i sends the vertical upper half-strip $\{0 < x < \pi, y > 0\}$ to the horizontal left half-strip $\{x < 0, 0 < y < \pi\}$. Thus the cosine function

$$z \mapsto \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

maps the vertical upper half-strip $\{0 < x < \pi, y > 0\}$ to the lower half-plane so that

(i) the horizontal line-segment $[0, \pi]$ goes to $[-1, 1]$,

(ii) the vertical line-segment $\{x = 0, 0 \leq y < \infty\}$ goes to $[1, \infty)$,

(iii) the vertical line-segment $\{x = \pi, 0 \leq y < \infty\}$ goes to $(-\infty, -1]$.

Let us now look at the sine function. We have $\sin z = -\cos\left(z + \frac{\pi}{2}\right)$ which maps the vertical upper half-strip $\left\{-\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0\right\}$ to the upper half-plane so that

(i) the horizontal line-segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ goes to $[-1, 1]$,

(ii) the vertical line-segment $\left\{x = -\frac{\pi}{2}, 0 \leq y < \infty\right\}$ goes to $(-\infty, -1]$,

(iii) the vertical line-segment $\left\{x = \frac{\pi}{2}, 0 \leq y < \infty\right\}$ goes to $[1, \infty)$.

Thus the temperature distribution T is given by $T = \frac{1}{2} + \frac{1}{\pi}\mathrm{Re}\,\sin^{-1} z$. We now use the formula for the real part of the inverse sine function to get

$$T = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\frac{1}{2}\left(\sqrt{(x+1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right),$$

where the branch $\sin^{-1} t$ of the inverse sine function of a real variable t is chosen to have the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

APPENDIX A: Hyperbola as Locus of a Point With Difference of Distances to Two Fixed Points Kept Constant

Let $c > a > 0$. Consider the locus of a point $P = (x, y)$ constrained by the condition that the distance between $P = (x, y)$ and $(c, 0)$ and the distance between $P = (x, y)$ and $(-c, 0)$ is always kept constant to be $2a$ or $-2a$. The equation of the locus of the point $P = (x, y)$ is given by

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Moving the second term on the left-hand side to the right-hand side and squaring both sides, we get

$$\left((x+c)^2 + y^2\right) = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + \left((x-c)^2 + y^2\right).$$

Moving the last term and the first term on the right-hand side to the right-hand side yields

$$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2}.$$

Dividing the equation by $4ac$ and squaring it yields

$$\left(\frac{x}{a} - \frac{a}{c}\right)^2 = \frac{(x-c)^2 + y^2}{c^2}.$$

After expanding the squares, we get

$$\frac{x^2}{a^2} - \frac{2x}{c} + \frac{a^2}{c^2} = \frac{x^2 - 2cx + c^2 + y^2}{c^2}$$

or

$$\frac{x^2}{a^2} + \frac{a^2}{c^2} = \frac{x^2}{c^2} + 1 + \frac{y^2}{c^2}.$$

Rearranging the terms, we get

$$\left(\frac{1}{a^2} - \frac{1}{c^2}\right)x^2 - \frac{y^2}{c^2} = 1 - \frac{a^2}{c^2}.$$

We let $b = \sqrt{c^2 - a^2}$ so that $b^2 = c^2 - a^2$ and we can rewrite the equation as

$$\frac{b^2}{a^2c^2}x^2 - \frac{y^2}{c^2} = \frac{b^2}{c^2}$$

which is equivalent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

APPENDIX B: Real Part of Arc Sine Computed by Characterization of Hyperbola as Locus of a Point With Difference of Distances to Two Fixed Points Kept Constant

Consider the function $z = \sin w$. We would like to compute the imaginary part of its inverse $w = \sin^{-1} z$ with a choice of a branch which we will specify later. First of all we observe that

$$\sin iy = \frac{1}{2i}\left(e^{i(iy)} - e^{-i(iy)}\right) = i\,\frac{1}{2}\left(e^{y} - e^{-y}\right) = i\ \sinh y$$

and

$$\cos iy = \frac{1}{2}\left(e^{i(iy)} + e^{-i(iy)}\right) = \frac{1}{2}\left(e^{y} - e^{-y}\right) = \cosh y.$$

As a consequence,

$$\begin{aligned} x + iy &= \sin w = \sin(u + iv) \\ &= \sin u \cos iv + \cos u \sin iv \\ &= \sin u \cosh v + i \cos u \sinh v. \end{aligned}$$

Separating the real parts and the imaginary parts, we get

$$\begin{cases} x = \sin u \cosh v \\ \\ y = \cos u \sinh v \end{cases}$$

Eliminating v by using the identity

$$\cosh^2 v - \sinh^2 v = 1,$$

we get

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1.$$

Let $a = \sin u$, $b = \cos u$ and $c = \sqrt{a^2 + b^2} = 1$. An equivalent description of this hyperbola is

$$\sqrt{(x+1)^2 + y^2} - \sqrt{(x+1)^2 + y^2} = 2 \sin u$$

which yields

$$u = \sin^{-1} \frac{1}{2}\left(\sqrt{(x+1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}\right).$$