General Variation Formula and Weierstrass-Erdmann Corner Condition

General Variation Formula. We take the variation of the functional

$$J = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx.$$

with the two end-points (x_1, y_1), (x_2, y_2) allowed freely to vary. Suppose we have a family curves $y = y(x, t)$ parametrized by t so that the extremal corresponds to $t = 0$. The new setting in our situation is that the end-points x_1 and x_2 are not independent of t. In fact, $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are both functions of t. So the functional J as a function of t can be written as

$$J(t) = \int_{x_1(t)}^{x_2(t)} F\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t)\right) dx.$$

Before we differentiate the functional $J(t)$ with respect to t, we first discuss the two variations of y with respect to t, one with x fixed and the other with x also varying as a function of t. By the chain rule applied to $y = y(x(t), t)$, we get

$$\frac{d}{dt} y(x(t), t) = \frac{\partial}{\partial t} y(x(t), t) + \frac{\partial}{\partial x} y(x(t), t) \frac{d}{dt} x(t).$$

We denote the left-hand side by δy, which represents the actual variation of an end-point so that if we require the end-point to lie on a prescribed curve it is this variation δy which will be used. The second factor $\frac{d}{dt} x(t)$ of the second term on the right-hand side is δx. The first term on the right-hand we will denote by $\partial_t y$ so that

$$(\dagger) \qquad \partial_t y = \delta y - y' \delta x.$$

By applying the Fundamental Theorem of Calculus to the variation of the upper limit and the lower limit of the integral functional, we get

$$(*) \qquad \delta J = F\left(x, y, y'\right) \delta x \Big|_{x=x_1}^{x=x_2} + \int_{x_1}^{x_2} \left(\partial_t F\left(x, y, y'\right)\right) dx.$$

The second term on the right-hand side can be dealt with in the same way as the simpler situation of the end-points being fixed and we obtain

$$\int_{x_1}^{x_2} \left(\partial_t F\left(x, y, y'\right)\right) dx = \int_{x_1}^{x_2} \left(F_y \partial_t y + F_{y'} \left(\partial_t y\right)'\right) dx$$

which by integration by parts gives

$$(\natural)\quad \int_{x_1}^{x_2} \left(F_y \partial_t y + F_{y'} \left(\partial_t y\right)'\right) dx = \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'}\right) \left(\partial_t y\right) dx + F_{y'} \partial_t y \Big|_{x=x_1}^{x=x_2}.$$

Now we combine together (†), (∗), and (♮) to get

$$\delta J = \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'}\right) \left(\partial_t y\right) dx + F_{y'} \delta y \Big|_{x=x_1}^{x=x_2} + \left(F - y' F_{y'}\right) \delta x \Big|_{x=x_1}^{x=x_2}.$$

This is known as the *general variation formula*.

Condition for End-Point to be on Prescribed Curve. Suppose the end-point (x_j, y_j) $(j = 1, 2)$ is constrained to be on the prescribed curve $g_j(x, y)$. First by considering the variation with the two end-points fixed we get the Euler-Lagrange equation to conclude from the general variation formula that

$$(\ddagger)\qquad \delta J = F_{y'} \delta y \Big|_{x=x_1}^{x=x_2} + \left(F - y' F_{y'}\right) \delta x \Big|_{x=x_1}^{x=x_2}.$$

By differentiating the equation $g_j(x_j, y_j) = 0$ with respect to the parameter t of the family of curves, we get

$$\frac{\partial g_j}{\partial x} \left(\delta x\big|_{x=x_j}\right) + \frac{\partial g_j}{\partial y} \left(\delta y\big|_{x=x_j}\right) = 0$$

which can be rewritten as

$$\left(\delta y\big|_{x=x_j}\right) = -\frac{\frac{\partial g_j}{\partial x}}{\frac{\partial g_j}{\partial y}} \left(\delta x\big|_{x=x_j}\right)$$

When we put this into (‡), from the vanishing of δJ we get

$$0 = F_{y'} \left(-\frac{\frac{\partial g_2}{\partial x}}{\frac{\partial g_2}{\partial y}} \delta x\Big|_{x=x_2}\right) - F_{y'} \left(-\frac{\frac{\partial g_1}{\partial x}}{\frac{\partial g_1}{\partial y}} \delta x\Big|_{x=x_1}\right) + \left(F - y' F_{y'}\right) \delta x \Big|_{x=x_1}^{x=x_2}.$$

Since δx is allowed to vary freely both at $x = x_1$ and $x = x_2$, it follows that

$$F_{y'} \left(-\frac{\frac{\partial g_j}{\partial x}}{\frac{\partial g_j}{\partial y}}\right) + \left(F - y' F_{y'}\right) = 0$$

at $x = x_j$ for $j = 1, 2$, which can be written as

$$\frac{\partial g_j}{\partial x} F_{y'} = \frac{\partial g_j}{\partial y} \left(F - y' F_{y'}\right)$$

at $x = x_j$ for $j = 1, 2$.

Weierstrass-Erdmann Corner Condition. Suppose our functional J is given as the sum of two integrals

$$J = \int_{x=x_1}^{x=\xi} F(x, y, y')\, dx + \int_{x=\xi}^{x=x_2} F(x, y, y')\, dx$$

so that the two end-points $y_1 = y(x_1)$ and $y_2 = y(x_2)$ but the ordinate $\eta = y(\xi)$ of the middle point with abscissa $x = \xi$ is allowed to vary freely. Then we can first fix the middle point and vary the two integrals separately to get one Euler-Lagrange equation on each of the two intervals $[x_1, \xi]$ and $[\xi, x_2]$. Then we consider the condition imposed by the free variation of the middle point (ξ, η) to get

$$\delta J = F_{y'} \delta y \Big|_{x=x_1}^{x=\xi} + (F - y' F_{y'})\, \delta x \Big|_{x=x_1}^{x=\xi} + F_{y'} \delta y \Big|_{x=\xi}^{x=x_2} + (F - y' F_{y'})\, \delta x \Big|_{x=\xi}^{x=x_2}$$

which, on account of the fixing of the two end-points (x_1, y_1) and (x_2, y_2), becomes

$$\delta J = \left(F_{y'} \Big|_{x=\xi-0}^{x=\xi+0} \right) \left(\delta y \Big|_{x=\xi} \right) + \left((F - y' F_{y'}) \Big|_{x=\xi+0}^{x=\xi-0} \right) \left(\delta x \Big|_{x=\xi} \right).$$

If both

$$\delta y \Big|_{x=\xi} \quad \text{and} \quad \delta x \Big|_{x=\xi}$$

are allowed to vary freely, the vanishing of δJ implies the following two statements which are known as the *Weierstrass-Erdmann corner condition*

$$F_{y'} \Big|_{x=\xi-0} = F_{y'} \Big|_{x=\xi+0},$$
$$(F - y' F_{y'}) \Big|_{x=\xi-0} = (F - y' F_{y'}) \Big|_{x=\xi+0}.$$

In other words, the two canonical coordinates $p = F_{y'}$ and $H = y' F_{y'} - F$ are continuous at the corner if the middle point is allowed to vary freely.

If we have the free variation of only $\delta y \Big|_{x=\xi}$, then we have only the partial Weierstrass-Erdmann corner condition

$$F_{y'} \Big|_{x=\xi-0} = F_{y'} \Big|_{x=\xi+0}.$$

On the other hand, If we have the free variation of only $\delta x\Big|_{x=\xi}$, then we have only the partial Weierstrass-Erdmann corner condition

$$\left(F - y' F_{y'}\right)\Big|_{x=\xi-0} = \left(F - y' F_{y'}\right)\Big|_{x=\xi+0}.$$

In general, if we have the free variation of only $\delta x\Big|_{x=\xi}$ and $\delta y\Big|_{x=\xi}$ on a prescribed curve $g(x, y) = 0$, then we have

$$\delta x\Big|_{x=\xi} g_x + \delta y\Big|_{x=\xi} g_y = 0$$

and the partial Weierstrass-Erdmann corner condition

$$-g_x F_{y'} + g_y \left(F - y' F_{y'}\right)\Big|_{x=\xi-0} = -g_x F_{y'} + g_y \left(F - y' F_{y'}\right)\Big|_{x=\xi+0}.$$

Snell's Law as Application of the Weierstrass-Erdmann Corner Condition. Consider the problem of light traveling in two different media where the velocity of light is c_1 and c_2 respectively. The light trajectory is the extremal of the functional

$$J = \int_{x=x_1}^{x=\xi} F_1\left(x, y, y'\right) dx + \int_{x=\xi}^{x=x_2} F_2\left(x, y, y'\right) dx,$$

where

$$F_j = \frac{\sqrt{1 + y'^2}}{c_j}$$

for $j = 1, 2$. First we consider the case of fixed the middle point (where the corner occurs) to get the two Euler-Lagrange equations

$$\left(F_j\right)_y - \frac{d}{dx}\left(F_j\right)_{y'} = 0.$$

Since each F_j is independent of x, we have the first integral which is the analog of the conservation of energy

$$H_j = y'\left(F_j\right)_{y'} - F_j = \text{constant}.$$

Explicit computation gives

$$H_j = y'\left(F_j\right)_{y'} - F_j = \frac{1}{c_j}\frac{y'^2}{\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{c_j} = \frac{-1}{c_j}\frac{1}{\sqrt{1+y'^2}}.$$

This means that light travels in a straight line in each of the two media. We now assume that the interface is given by $x = \xi$ with ξ fixed. Then only δy is allowed to vary freely at $x = \xi$. At the interface, the canonical coordinate p is continuous. That is,

$$(F_1)_{y'}\Big|_{x=\xi-0} = (F_2)_{y'}\Big|_{x=\xi+0},$$

which means

$$\frac{1}{c_1}\frac{y'}{\sqrt{1+y'^2}}\Bigg|_{x=\xi-0} = \frac{1}{c_2}\frac{y'}{\sqrt{1+y'^2}}\Bigg|_{x=\xi+0}.$$

Introduce the angle of incidence θ_1 and the angle of refraction θ_2 given by

$$y'\Big|_{x=\xi-0} = \tan\theta_1 \quad \text{and} \quad y'\Big|_{x=\xi+0} = \tan\theta_2.$$

Then we can write the above condition in the form of Snell's law for the refraction of light which states that

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$