Homework Assigned on December 7, 2006
due December 19, 2006

Problem 1 (#18 on Page 52 of Gelfand and Fomin). Find the extremals of the functional

$$J[y]=\int_0^1\left(y'^2+x^2\right)dx,$$

subject to the conditions

$$y(0)=0,\quad y(1)=0,\quad \int_0^1 y^2dx=2.$$

Problem 2 (#1 on Page 94 of Gelfand and Fomin). Use the canonical differential equations to find the extremals of the functional

$$\int\sqrt{x^2+y^2}\,\sqrt{1+y'^2}\,dx,$$

and verify that they are of the form

$$x^2\cos\alpha+2xy\sin\alpha-y^2\cos\alpha=\beta,$$

where α and β are constants.

Hint: The Hamiltonian is

$$H(x,y,p)=-\sqrt{x^2+y^2-p^2},$$

and the corresponding canonical system

$$\frac{dp}{dx}=\frac{y}{\sqrt{x^2+y^2-p^2}},\quad \frac{dy}{dx}=\frac{p}{\sqrt{x^2+y^2-p^2}}$$

has the first integral

$$p^2-y^2=C^2,$$

where C is a constant.

Problem 3 (#5 on Page 95 of Gelfand and Fomin). Consider the variation problem for functional

$$J[r,\theta]=\int_{t_0}^{t_1}\left[\frac{m}{2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r}\right]dt,$$

which comes from the problem of the plane motion of a particle of mass m attracted to the origin by a force $\frac{k}{r^2}$. For this variation problem the radius vector r and the angle θ are the dependent variables and the time t is the independent variable. The symbol $\dot{r}$ denotes the derivative of r with respect to t and the symbol $\dot{\theta}$ denotes the derivative of θ with respect to t.

Verify that the functional $J[r,\theta]$ is invariant under rotations (which are given by $\theta\mapsto\theta+\alpha$ with r unchanged and $\alpha\in\mathbb{R}$). Use Noether's theorem (in polar coordinates) to verify Kepler's law that the line segment joining the particle to the origin sweeps out equal areas in equal times.

Problem 4 (#7 on Page 95 of Gelfand and Fomin). Write and solve the Hamilton-Jacobi partial differential equation corresponding to the functional

$$J[y]=\int_{x_0}^{x_1} f(y)\sqrt{1+y'^2}\,dx,$$

and use the result to find the extremals of $J[y]$.

Hint: The Hamilton-Jacobi partial differential equation is

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2=f^2(y),$$

with solution

$$S=\alpha x+\int_{y_0}^{y}\sqrt{f^2(\eta)-\alpha^2}\,d\eta+\beta.$$

The extremals are

$$x-\alpha\int_{y_0}^{y}\frac{d\eta}{\sqrt{f^2(\eta)-\alpha^2}}=\text{constant}.$$

Problem 5 (#11 on Page 130 of Gelfand and Fomin). Prove that the extremal

$$y=\frac{y_1 x}{x_1}$$

corresponds to a local minimum of both functionals

$$\int_0^{x_1} \frac{dx}{y'}, \quad \int_0^{x_1} \frac{dx}{y'^2},$$

where $y(0) = 0$, $y(x_1) = y_1$, $x_1 > 0$, $y_1 > 0$.

Problem 6 (#15 on Page 130 of Gelfand and Fomin). Consider the catenary

$$y = c \cosh\left(\frac{x+b}{c}\right)$$

(where b and c are constants), which is an extremal for the variational problem of the functional

$$J = \int_{x_0}^{x_1} y\sqrt{1+y'^2}\, dx.$$

Show that any point on the catenary except the vertex $(-b, c)$ has one and only one conjugate point, and show that the tangents to any pair of conjugate points intersect on the x-axis.