Homework Assigned on November 16, 2006
due November 28, 2006

Problem 1 (Poisson's Integral Formula for the Exterior of a Circle [#4 on p.167 of Strauss]). Suppose u is a *bounded* twice continuously differentiable function on the exterior $\{ x^2 + y^2 > a^2 \}$ of the circle C_a of radius $a > 0$ centered at the origin. Assume that u is continuous up to the circle C_a and let $h(\theta) = u\left(e^{i\theta}\right)$ for $0 \leq \theta \leq 2\pi$. Derive the following Poisson's integral formula for the exterior of the circle C_a.

$$u\left(re^{i\theta}\right) = \left(r^2 - a^2\right) \int_{\varphi=0}^{2\pi} \frac{h\left(\varphi\right)}{a^2 - 2ar\cos\left(\theta - \varphi\right) + r^2} \frac{d\varphi}{2\pi}$$

for $r > a$.

Hint: Imitate the argument used in the derivation of the Poisson integral formula by Fourier series and convolution.

Problem 2 (Uniqueness of Harmonic Solution with the Robin Condition [#11 on p.168 of Strauss]). The boundary condition for solving the Laplace equation u on a bounded domain D is called the Dirichlet condition if the value of u is specified at the boundary ∂D of D. The boundary condition for solving the Laplace equation u on a bounded domain D is called the Neumann condition if the outward normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ of u is specified at the boundary ∂D of D. The boundary condition for solving the Laplace equation u on a bounded domain D is called the Robin condition if the normal derivative $\frac{\partial u}{\partial \mathbf{n}} + au$ is specified at the boundary ∂D of D, where a is a function on the boundary ∂D of D.

Prove the uniqueness of the Robin problem

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial \mathbf{n}} + au = 0 \text{ on } \partial D,$$

where D is any domain in three dimensions and where a is a positive constant.

Hint: Imitate the argument used in the proof of the uniqueness for the Neumann boundary value problem from the divergence theorem.

Problem 3 (Hopf Lemma [#12 on p.168 of Strauss]). (a) Let $0 < a < b$. Let $G_{a,b}$ be the domain defined by $a < r < b$, where $r = |\mathbf{x}|$ and $\mathbf{x}$ is a point in $\mathbb{R}^2$ (or in general in $\mathbb{R}^n$). Let $\sigma > 0$ and let $v_\sigma\left(\mathbf{x}\right) = e^{-\sigma r^2} - e^{-\sigma b^2}$. Verify that there exists some $\sigma_0 > 0$ such that for $\sigma \geq \sigma_0$ one has $\Delta v_\sigma > 0$ on $\overline{G_{a,b}}$, where Δv_σ is the Laplacian of v_σ and $\overline{G_{a,b}}$ is the closure of $G_{a,b}$. Verify also that $\frac{\partial}{\partial \mathbf{n}} v_\sigma < 0$ at every point of $r = b$, where $\frac{\partial}{\partial \mathbf{n}}$ means differentiation in the direction of the outward normal $\mathbf{n}$ of $G_{a,b}$.

(b) Suppose w is a twice continuous differentiable real-valued function on some open neighborhood of the origin such that it assumes its local maximum at the origin. Verify that the Laplacian of w at the origin must be ≤ 0.

Hint: Use the second derivative test.

(c) Prove the following strong form of the maximum principle, called the Hopf form of the maximum principle. If $u(\mathbf{x})$ is a nonconstant harmonic function on a domain D that has a maximum at $\mathbf{x}_0$ (necessarily on the boundary ∂D of D), then $\frac{\partial u}{\partial \mathbf{n}} > 0$ at $\mathbf{x}_0$, where $\frac{\partial u}{\partial \mathbf{n}}$ is the *outward* normal derivative of u.

Hint: Without loss of generality we can assume that $G_{a,b}$ from Part (a) is contained in D so that $\{r = b\}$ touches ∂D at $\mathbf{x}_0$. Verify that there exists some $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and for $\sigma \geq \sigma_0$ one has

(i) $\Delta \left(u - u(\mathbf{x}_0) + \varepsilon v_\sigma\right) > 0$ on $G_{a,b}$,

(ii) $u - u(\mathbf{x}_0) + \varepsilon v_\sigma \leq 0$ on $\partial G_{a,b} = \{r = a\} \cup \{r = b\}$ and hence also on all of $G_{a,b}$ by (b) and (i),

(iii) $\frac{\partial u}{\partial \mathbf{n}} \geq -\varepsilon \frac{\partial v_\sigma}{\partial \mathbf{n}}$ at $\mathbf{x}_0$ from (ii).

Problem 4. Let $-\infty < x_1 < x_2 < \infty$ and $y_1, y_2 \in \mathbb{R}$. Find the extremals $y = y(x)$ for the following functionals with $y(x_1) = y_1$ and $y(x_2) = y_2$.

(*a*) $$\int_{x_1}^{x_2} \sqrt{y\left(1 + y'^2\right)}\, dx.$$

(*b*) $$\int_{x_1}^{x_2} \frac{1 + y^2}{y'^2}\, dx.$$

(*c*) $$\int_{x_1}^{x_2} y'\left(1 + x^2 y'\right) dx.$$

(*d*) $$\int_{x_1}^{x_2} \left(y^2 + y'^2 - 2y \sin x\right) dx.$$