Homework Assigned on November 30, 2006
due December 12, 2006

Problem 1 (#5 on p.64 of Gelfand and Fomin). Find the curves for which the functional

$$J[y] = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\, dx, \quad y(0) = 0$$

can have an extrema if

(a) the point (x_1, y_1) can vary along the line $y = x - 5$;

(b) the point (x_1, y_1) can vary along the circle $(x-9)^2 + y^2 = 9$.

Answer: (a) $y = \pm\sqrt{10x - x^2}$; (b) $y = \pm\sqrt{8x - x^2}$.

Problem 2 (#10 on p.64 of Gelfand and Fomin). Find the curves for which the functional

$$J[y, z] = \int_0^{x_1} \left(y'^2 + z'^2 + 2yz\right) dx$$

can have extrema, given that $y(0) = z(0) = 0$, while the point (x_1, y_1, z_1) can vary in the plane $x = x_1$.

Problem 3 (#12 on p.65 of Gelfand and Fomin). Find the curves for which the functional

$$J[y] = \int_0^1 \left(1 + y''^2\right) dx$$

can have extrema, given that $y(0) = 0$, $y'(0) = 1$, $y(1) = 1$, while $y'(1)$ can vary arbitrarily.

Problem 4 (#15 on p.65 of Gelfand and Fomin). Prove that the functional

$$J[y] = \int_{x_0}^{x_1} \left(ay'^2 + byy' + cy^2\right) dx, \quad y(x_0) = y_0, \quad y(x_1) = y_1,$$

where $a \neq 0$, can have no broken extremals.

Problem 5 (#21 on p.66 of Gelfand and Fomin). Find the curves for which the functional

$$J[y] = \int_0^{10} y'^3 dx, \quad y(0) = 0, \quad y(10) = 0$$

can have extrema, given that the admissible curves cannot penetrate the interior of the circle with equation

$$(x-5)^2 + y^2 = 9.$$

Answer:

$$y = \begin{cases} \pm\frac{3}{4}x & \text{for} \quad 0 \leq x \leq \frac{16}{5}, \\ \pm\sqrt{9-(x-5)^2} & \text{for} \quad \frac{16}{5} \leq x \leq \frac{34}{5}, \\ \mp\frac{3}{4}(x-10) & \text{for} \quad \frac{34}{5} \leq x \leq 10. \end{cases}$$