Homework Assigned on November 9, 2006
due November 28, 2006

Problem 1 (Poisson's Integral Formula for the Upper Half-Plane [#1 on p.171 of Ahlfors]). Verify Part (a) by imitating the derivation of the Poisson integral formula for the unit disk by the method of the argument function and plane Euclidean geometry and by using an $\mathbb{R}$-linear combination of the constant function 1 and the two functions

$$\arg(z-\alpha)\,, \quad \arg(z-\beta)$$

for $-\infty < \alpha < \beta < \infty$ and passing to the limit of its quotient by $\beta - \alpha$ as $\beta \to \alpha$.

(a) Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2}\, U(\xi)\, d\xi$$

represents a harmonic function in the upper half-plane with boundary value $U(\xi)$ at points of continuity.

(b) Verify Part (a) by using the Poisson integral formula for the unit disk and a linear fractional transformation which maps conformally the unit disk onto the upper half-plane.

Problem 2 (Pointwise Convergence of the Cesàro Sum of the Fourier Series of a Continuous Periodic Function). Let $f(x)$ be a continuous function on $\mathbb{R}$ with period 2π. Let s_n be the n-th partial sum of the Fourier series of $f(x)$ (in terms of the cosine and sine functions) so that

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx,$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx$$

are by definition the Fourier coefficients of $f(x)$. Let σ_n denote the n-th Cesàro sum of the Fourier series of $f(x)$ defined by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n}.$$

Let $D_n(x)$ denote the n-th Dirichlet-Dini kernel which is defined by

$$D_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi} \frac{\sin\left(n + \frac{1}{2}\right) x}{\sin \frac{x}{2}}.$$

Define the Féjer kernel $F_n(x)$ by

$$F_n(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{n-1}(x)}{n}.$$

(a) Verify that $\sigma_n = f * F_n$. Here the notation $f * F_n$ denotes the convolution of the two functions $f(x)$ and $F_n(x)$ defined by

$$\left(f * F_n\right)(x) = \int_{y=-\pi}^{\pi} f(x-y) F_n(y) dy.$$

(b) Verify that

$$F_n(x) = \frac{1}{2\pi n} \frac{\sin^2 \frac{nx}{2}}{\sin^2 \frac{x}{2}}.$$

(c) Verify that the family of functions $\{F_n(x)\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ with period 2π is an approximate identity on $[-\pi, \pi]$ in the sense that the following three conditions are satisfied.

(i) *(Nonnegativity)* $F_n(x) \geq 0$ for $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$.

(ii) *(Unit Integral)* $\int_{-\pi}^{\pi} F_n(x)\, dx = 1$ for all $n \in \mathbb{N}$.

(iii) *(Integral Outside Any Neighborhood of the Origin Approaching 0)* For any $\eta > 0$ the integral

$$\int_{\substack{-\pi \leq x \leq \pi \\ |x| \geq \eta}} F_n(x)\, dx$$

approaches 0 as $n \to \infty$.

(d) Use Part (c) to show that $\lim_{n\to\infty} \sigma_n(x) = f(x)$. This kind of convergence of the average of partial sums is known as the convergence of the Cesàro sum or convergence in the sense of Cesàro.

Problem 3 (Solution of the Laplace Equation on the Unit Disk with Neumann Boundary Condition). For a complex variable ζ and another complex variable z with $|z| < |\zeta|$, the gradient of $\log|z-\zeta|$ with respect to z is equal to the negative of the gradient of $\log|z-\zeta|$ with respect to ζ. Moreover, by the Cauchy-Riemann equations for the variable ζ, the outward radial directive of $\log|z-\zeta|$ with respect to ζ is equal to the tangential derivative of a suitably defined branch of $\arg(z-\zeta)$ along the circle of radius $|\zeta|$ centered at the origin in the counterclockwise sense. On the other hand, for $|\zeta| = 1$ the tangential derivative of $\arg(z-\zeta)$ with respect to ζ along the unit circle is related to

$$\lim_{\beta\to\alpha} \frac{1}{\beta-\alpha} \arg \frac{z-e^{i\beta}}{z-e^{i\alpha}},$$

which, in turn, is related to the Poisson integral kernel constructed by the method of the argument function and plane Euclidean geometry. This observation motivates the following way of constructing a solution of the Laplace equation on the unit disk with Neumann boundary condition.

For z inside the unit disk and for ζ on the unit circle, let

$$Q(z,\zeta) = -2\log|z-\zeta|.$$

Let $g(\zeta)$ be a continuous function on the unit circle $|\zeta| = 1$ such that

$$(*) \qquad \int_{\varphi=-\pi}^{\pi} g\left(e^{i\varphi}\right) d\varphi = 0.$$

For $z \in \mathbb{C}$ with $|z| < 1$, let

$$u(z) = \frac{1}{2\pi} \int_{\varphi=-\pi}^{\pi} Q\left(z, e^{i\varphi}\right) g\left(e^{i\varphi}\right) d\varphi.$$

Verify that $u(z)$ is harmonic on the open unit disk $\{|z| < 1\}$ and that for every fixed $\varphi \in \mathbb{R}$ the radial derivative

$$\frac{\partial}{\partial r} u\left(re^{i\varphi}\right)$$

of u approaches $g\left(e^{i\varphi}\right)$ as $r \to 1$ from $r < 1$.

Note that the function $g(\zeta)$ which defines the Neumann boundary condition must satisfy $(*)$ because of the divergence theorem for the gradient of the harmonic function u on the unit disk.