Homework Assigned on October 12, 2006
due October 24, 2006

Problem 1. (a) Verify the following infinite partial fraction decomposition

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+4n^2\pi^2}.$$

(b) Verify the following infinite partial fraction decomposition

$$\tan z=2z\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2\pi^2-z^2}.$$

(c) Verify the following infinite product expansion

$$\cos z=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right).$$

Problem 2. Sum the following two series $\sum f(n)$ by considering the contour integral $\int \pi f(z)\cot \pi z dz$.

$$\sum_{n=1}^{\infty}\frac{1}{n^4+a^4}, \qquad \sum_{n=1}^{\infty}\frac{n^2}{n^4+a^4},$$

where a is a positive number.

Problem 3. Let D be the domain in $\mathbb{C}$ which is the open strip $\{\,0<y<\pi\,\}$ minus the ray $\left\{\,x\leq 0,\ y=\frac{\pi}{2}\,\right\}$. Find a harmonic function u on D such that the boundary value of u on $\left\{\,x\leq 0,\ y=\frac{\pi}{2}\,\right\}$ is -1 and the boundary value of u on both $\{\,y=0\,\}$ and $\{\,y=\pi\,\}$ is 1. *Hint:* consider first the exponential map $z\mapsto e^z$.

Problem 4. Let C_1 be the circle $|z|=5$ and C_2 be the circle $|z-2|=2$. Let D be the domain inside the circle C_1 and outside the circle C_2. Find a harmonic function u on D whose boundary value at C_1 is 1 and whose boundary value at C_2 is 0. *Hint:* consider the linear fractional transformation with real coefficients which maps the four points $-5,0,4,5$ to the four points $-R,-1,1,R$ with R to be determined.

Problem 5 (from Ahlfors p.83, #6). Suppose that a linear fractional transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same.