Homework Assigned on October 19, 2006
due October 31, 2006

Problem 1. Let T denote the bounded steady temperature on the first quadrant $\{x > 0, y > 0\}$ with the following three constraints:

(i) The boundary value of T on $\{x > 1, y = 0\}$ is 1.

(ii) The boundary value of T on $\{x = 0, y > 1\}$ is 0.

(iii) Both the boundary-segment $\{x = 0, 0 < y < 1\}$ and the boundary-segment $\{0 < x < 1, y = 0\}$ are insulated.

Verify that

$$T = \frac{1}{2} + \frac{1}{\pi} \sin^{-1} \frac{1}{2} \left(\sqrt{(x^2 - y^2 + 1)^2 + 4x^2y^2} - \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2} \right),$$

where the range of the inverse sine function is chosen to be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Problem 2. Let $H > 0$ and $A > 0$. Let D be the domain obtained from the upper half-plane $\{y > 0\}$ by removing the line-segment $\{x = 0, 0 < y \leq H\}$. Consider the 2-dimensional incompressible irrotational fluid flow in D whose velocity at infinity is horizontal from left to right with speed A. Find the velocity of the flow at the point $(x, y) \in D$.

(The practical interpretation of the problem is that there is a dam of height H and the flow is infinitely deep with velocity A at infinity.)

Problem 3. Let D be the domain obtained from $\{0 < y < \pi\}$ by removing the line-segment joining the origin to $\left(0, \frac{\pi}{2}\right)$. Find an orientation-preserving conformal mapping F which maps D to the upper half-plane.

Hint: First apply the map $z \mapsto e^z$. Then apply the map $z \mapsto \frac{z-1}{z+1}$.

Problem 4. Let D be as in Problem 3. Consider the 2-dimensional incompressible irrotational fluid flow in D whose velocity at infinity is horizontal from left to right with speed A. Use the result in Problem 3 to find the velocity of the flow at the point $(x, y) \in D$.

(The practical interpretation of the problem is for a flow in a very wide channel of height π with an obstructive wall of height $\frac{\pi}{2}$.)

Problem 5. Let D be the domain obtained from $\mathbb{C}$ by removing the two line segments $\{x = 0, 1 \leq y < \infty\}$ and $\{x = 0, -\infty < y \leq -1\}$. Consider a flow in D from left to right whose velocity at infinity is horizontal. Assume that the net rate of the flow across $\{x = 0, -1 < y < 1\}$ is A. Find the velocity of the flow at the point $(x, y) \in D$.

Hint: The net rate A of the flow across $\{x = 0, -1 < y < 1\}$ is equal to the difference between the constant value of the streamline function on $\{x = 0, 1 \leq y < \infty\}$ and the constant value of the streamline function on $\{x = 0, -\infty < y \leq -1\}$, because A is the integral over $\{x = 0, -1 < y < 1\}$ of the horizontal component of the flow velocity and one can apply the Cauchy-Riemann equation and the Fundamental Theorem of Calculus.