Homework Assigned on October 26, 2006
due November 7, 2006

Problem 1 (fluid flow in a channel through a slit). Let Q be a positive number. (a) Find a bounded harmonic function u on the upper half-plane so that

(i) its boundary value on $-\infty < x < 0$ is $\frac{Q}{2}$,

(ii) its boundary value on $0 < x < 1$ is Q, and

(iii) its boundary value on $1 < x < \infty$ is 0.

Hint: Use the construction technique for the integrand of a Schwarz-Christoffel transformation.

(b) Find a bounded harmonic function φ on the strip $\{\, 0 < y < \pi \,\}$ such that

(i) its boundary value on $\{\, y = \pi \,\}$ is $\frac{Q}{2}$,

(ii) its boundary value on $\{\, x < 0, y = 0 \,\}$ is Q, and

(iii) its boundary value on $\{\, x > 0, y = 0 \,\}$ is 0.

Hint: Find φ as the function u in (a) composed with a linear fractional transformation and the exponential function.

(c) Show that the curves $\{\varphi(x, y) = \text{constant}\}$ are given by $\tan\frac{y}{2} = c \tanh\frac{x}{2}$ for some constant c. *Hint:* Use $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$.

(d) Interpret (b) and (c) as solving the following problem of fluid flow. There is a 2-dimensional steady irrotational incompressible fluid flow of constant density in the channel represented by the strip $\{\, 0 < y < \pi \,\}$. The fluid enters through a slit represented by the origin at the rate of Q units per unit time so that the flow exits at each end of the channel (represented by $x = -\infty$ and $x = \infty$) at the rate of $\frac{Q}{2}$ units per unit time. Show that the equation of a streamline is given by $\tan\frac{y}{2} = c \tanh\frac{x}{2}$ for some constant c.

Problem 2 (elliptic integrals and conformal mapping from the upper half-plane to a rectangle). Let $0 < k < 1$. Let

$$K = \int_{t=0}^{1} \frac{dt}{\sqrt{(1 - t^2)\,(1 - k^2 t^2)}},$$

$$iK' = \int_{t=1}^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Let R be the open rectangle with vertices at

$$K, \quad K+iK', \quad -K+iK', \quad -K.$$

Let $g(\zeta)$ be a branch for the function

$$\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}$$

on the upper half-plane. Consider the Schwarz-Christoffel transformation

$$w = \int_0^z \frac{d\zeta}{g(\zeta)}.$$

(a) Verify that for some choice of the branch $g(\zeta)$ the Schwarz-Christoffel transformation maps the upper half-plane in the z variable one-one onto the rectangle R in the w variable.

(b) Describe how that particular branch of $g(\zeta)$ is defined (*i.e.,* what cuts have to be made in $\mathbb{C}$ and what the ranges of the numerical values of the angles in polar representations are).

(c) Describe the correspondence between the quadruple $\left\{1, -1, \frac{1}{k}, -\frac{1}{k}\right\}$ of points in the z variable and the four vertices of R in the w variable (*i.e.,* which point in the quadruple goes to which vertex of R).

The definite integrals K and K' are known as *complete elliptic integrals of the first kind.*

Problem 3. Let $h > 0$. Let Ω be the domain in $\mathbb{C}$ with variable $w = u + iv$ obtained by removing the rectangle $\{\, 0 < v \leq h,\ u \leq 0 \,\}$ from the open upper half-plane $\{\, v > 0 \,\}$. Consider the Schwarz-Christoffel transformation from the open upper half-plane in the z variable to the domain Ω in the w variable whose derivative $\frac{dw}{dz}$ is given by

$$\frac{dw}{dz} = A\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}},$$

where A is a nonzero complex number. Verify that for some nonconstant complex number A the Schwarz-Christoffel transformation can be written in the following form

$$w=\frac{h}{\pi}\left((z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}+\log\left(z+(z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}\right)\right),$$

where

(i) the branch of $(z+1)^{\frac{1}{2}}$ is chosen with $0 \leq \arg(z+1) \leq \pi$,

(ii) the branch of $(z-1)^{\frac{1}{2}}$ is chosen with $0 \leq \arg(z-1) \leq \pi$, and

(iii) the branch of log is the principal branch with the argument defined between $-\pi$ and π.

Moreover, verify that that particular Schwarz-Christoffel transformation maps

(i) the interval $(-\infty, -1]$ in the z variable to the line-segment

$$\{-\infty < u \leq 0,\ v = h\}$$

in the w variable.

(ii) the interval $[-1, 1]$ in the z variable to the line-segment

$$\{u = 0,\ 0 \leq v \leq h\}$$

in the w variable, and

(iii) the interval $[1, \infty)$ in the z variable to the line-segment $[0, \infty)$ in the w variable.

Problem 4. Show that, if α and $\beta \neq 0$ are real numbers, the equation

$$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$$

has $n-1$ roots with positive real parts if n is odd, and n roots with positive real parts if n is even.

Hint: Apply the argument principle to the right half-disk of radius R and let $R \to \infty$.