Remarks on Solution of Problem 2 of Homework Assigned on October 26 2006 due November 7, 2006

Problem 2 (elliptic integrals and conformal mapping from the upper half-plane to a rectangle). Let $0 < k < 1$. Let

$$K = \int_{t=0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$K' = \int_{t=1}^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Let R be the open rectangle with vertices at

$$K, \quad K + iK', \quad -K + iK', \quad -K.$$

Let $g(\zeta)$ be a branch for the function

$$\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}$$

on the upper half-plane. Consider the Schwarz-Christoffel transformation

$$w = \int_0^z \frac{d\zeta}{g(\zeta)}.$$

(a) Verify that for some choice of the branch $g(\zeta)$ the Schwarz-Christoffel transformation maps the upper half-plane in the z variable one-one onto the rectangle R in the w variable.

(b) Describe how that particular branch of $g(\zeta)$ is defined (*i.e.,* what cuts have to be made in $\mathbb{C}$ and what the ranges of the numerical values of the angles in polar representations are).

(c) Describe the correspondence between the quadruple $\left\{1, -1, \frac{1}{k}, -\frac{1}{k}\right\}$ of points in the z variable and the four vertices of R in the w variable (*i.e.,* which point in the quadruple goes to which vertex of R).

Remarks on Solution. The function $g(\zeta)$ can be written as

$$g(\zeta) = k\left(\zeta - \left(-\frac{1}{k}\right)\right)^{\frac{1}{2}} (\zeta - (-1))^{\frac{1}{2}} (\zeta - 1)^{\frac{1}{2}} \left(\zeta - \frac{1}{k}\right)^{\frac{1}{2}}.$$

Let

$$\zeta - \left(-\frac{1}{k}\right) = r_{-\frac{1}{k}} e^{i\theta_{-\frac{1}{k}}},$$
$$\zeta - (-1) = r_{-1} e^{i\theta_{-1}},$$
$$\zeta - 1 = r_1 e^{i\theta_1},$$
$$\zeta - \frac{1}{k} = r_{\frac{1}{k}} e^{i\theta_{\frac{1}{k}}}.$$

For Part (b), the cuts will be given by

$$0 \le \theta_{-\frac{1}{k}} \le \pi,$$
$$0 \le \theta_{-1} \le \pi,$$
$$2\pi \le \theta_1 \le 3\pi,$$
$$0 \le \theta_{\frac{1}{k}} \le \pi.$$

Note the special choice of the range for the numerical value for the angle θ_1 which is different from the other three. The reason is that we have to make sure that when $\zeta \in (0,1)$ the value of $g(\zeta)$ should be positive. For our choice of the four ranges, when $\zeta \in (0,1)$ we get

$$\theta_{-\frac{1}{k}} = 0,$$
$$\theta_{-1} = 0,$$
$$\theta_{-1} = 3\pi,$$
$$\theta_{\frac{1}{k}} = \pi$$

and

$$g(\zeta) = \sqrt{r_{-\frac{1}{k}} r_{-1} r_1 r_{\frac{1}{k}}}\, e^{\frac{i}{2}\left(\theta_{-\frac{1}{k}} + \theta_{-1} + \theta_1 + \theta_{\frac{1}{k}}\right)}$$
$$= \left|\zeta - \left(-\frac{1}{k}\right)\right|^{\frac{1}{2}} |\zeta - (-1)|^{\frac{1}{2}} |\zeta - 1|^{\frac{1}{2}} \left|\zeta - \frac{1}{k}\right|^{\frac{1}{2}} e^{i\,2\pi} > 0.$$