Homework Assigned on October 5, 2006
due October 17, 2006

Problem 1 (partly from Ahlfors p.161, #3). Evaluate the following integrals by the method of residues:

(a) $$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$$

(b) $$\int_{0}^{\infty} \frac{x \sin x \, dx}{(x^2 + a^2)^2} \quad (a \in \mathbb{R})$$

(c) $$\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

(*Hint:* Convert $\sin^2 x$ to $\cos 2x$ first.)

Problem 2 (partly from Ahlfors p.161, #3). Evaluate the following integrals by applying the method of residues to branches of holomorphic functions.

(a) $$\int_{0}^{\infty} \frac{x^{\frac{1}{3}}}{1 + x^2}\, dx$$

(b) $$\int_{0}^{\infty} \frac{\log\left(1 + x^2\right)\, dx}{x^{1+\alpha}} \quad (0 < \alpha < 2)$$

(*Hint:* Try integration by parts.)

(c) $$\int_{0}^{\infty} \frac{(\log x)^2\, dx}{1 + x^2}$$

Problem 3. Use the branch cut $[0, 1]$ and the theory of residues to establish the following formula of integration.

$$\int_{0}^{1} \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}\, dx = \frac{3\pi \sqrt[4]{2}}{64}.$$

Hint: Follow the techniques used for the in-class computation of

$$\int_{0}^{1} \frac{dx}{x^{\alpha}(1-x)^{1-\alpha}} = \frac{\pi}{\sin \alpha\pi} \quad (0 < \alpha < 1).$$

Problem 4. Let D be the domain in $\mathbb{C}$ formed from $\mathbb{C}$ by deleting the three line-segments $[-1, i]$, $[1, i]$ and $\{x = 0,\ y \geq 1\}$. For $a \in D$ let $f(a)$ be defined by

$$f(a) = \int_{C_a} \frac{-2z}{1 - z^2}\, dz,$$

where C_a is any piecewise smooth curve in D joining 0 to a.

(a) Show by Cauchy's theorem for holomorphic functions that the definition of $f(a)$ is independent of the choice of the piecewise smooth curve C_a in D.

(b) Show that $f(z)$ is a branch of $\log(1 - z^2)$ on D in the sense that $f(z)$ is holomorphic on D and $e^{f(z)} = 1 - z^2$ for $z \in D$.

(c) Compute $f(2)$.