**Homework on Partial Differential Equations**
**(No Need to Hand In)**

*Problem 1 (#2 on Page 256 of Strauss).* Determine the vibrations

$$u_{tt} = c^2 \left(u_{xx} + u_{yy}\right)$$

of a circular drumhead of radius $a$ (held fixed on the boundary) with the initial conditions

$$u = 1 - \frac{r^2}{a^2} \quad \text{when} \quad t = 0$$

and $u_t \equiv 0$ when $t = 0$. (The answer is given on Page 410 of Strauss.)

*Problem 2 (#4 on Page 263 of Strauss).* Solve the wave equation

$$u_{tt} - c^2 \Delta u = 0$$

on the ball $\{ r < a \}$ of radius $a$ in $\mathbb{R}^3$ with the conditions

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad \{ r = a \},$$

$$u = z = r \cos\theta \quad \text{when} \quad t = 0,$$

$$u_t \equiv 0 \quad \text{when} \quad t = 0,$$

where $\theta$ is the colatitude and $\varphi$ is the longitude in the spherical coordinates $(r, \theta, \varphi)$ of $\mathbb{R}^3$ so that $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, and $z = r \cos\theta$. (The answer is given on Page 410 of Strauss.)

*Problem 3 (#18 on Page 274 of Strauss).* Find an equation for the eigenvalues $\lambda$ and find the eigenfunctions $v$ of the negative Laplacian $-\Delta$ in the disk $\{ x^2 + y^2 < a^2 \}$ for eigenvalues $\lambda$ (in the sense that $-\Delta v = \lambda v$) with the boundary condition

$$\frac{\partial v}{\partial r} + hv = 0$$

on the circle $\{ x^2 + y^2 = a^2 \}$, where $h$ is a constant. (The answer is given on Page 411 of Strauss.)

*Problem 4 (#1 and #4 on Page 278 of Strauss).* Consider the $\ell$-th Legendre polynomial

$$P_\ell(z) = \frac{1}{2^\ell} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(2\ell - 2j)!}{(\ell - 2j)!(\ell - j)!} z^{\ell - 2j},$$

where $m = \frac{\ell}{2}$ if $\ell$ is even, and $m = \frac{\ell-1}{2}$ if $\ell$ is odd.

(a) Prove the following recurrent relation for Legendre polynomials

$$(\ell + 1)P_{\ell+1}(z) - (2\ell + 1)zP_\ell(z) + \ell P_{\ell-1}(z) = 0.$$

(b) Show that

$$\int_{x=-1}^{1} x^2 \, P_\ell(x) \, dx = 0$$

for $\ell \geq 3$.

*Problem 5 (#5 on Page 278 of Strauss).* Let $f(x) = x$ for $0 \leq x < 1$, and $f(x) = 0$ for $-1 < x \leq 0$. Find the coefficients $a_\ell$ in the expansion $f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$ of $f(x)$ in the interval $(-1, 1)$ in terms of Legendre polynomials

$$P_\ell(z) = \frac{1}{2^\ell} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(2\ell - 2j)!}{(\ell - 2j)!(\ell - j)!} z^{\ell - 2j},$$

where $m = \frac{\ell}{2}$ if $\ell$ is even, and $m = \frac{\ell-1}{2}$ if $\ell$ is odd. (The answer is given on Page 411 of Strauss.)

*Problem 6 (#7 on Page 278 of Strauss).* Find the harmonic function in the ball $\{ x^2 + y^2 + z^2 < a^2 \}$ with the boundary condition $u = A$ on the top hemisphere $\{ x^2 + y^2 + z^2 = a^2, \, z > 0 \}$ and with $u = B$ on the bottom hemisphere $\{ x^2 + y^2 + z^2 = a^2, \, z < 0 \}$, where $A$ and $B$ are constants. (The answer is given on Page 411 of Strauss.)