Homework Assigned on September 21, 2006
due September 28, 2006

Problem 1 (from Ahlfors p.2, #1 and p.4, #1). Compute the real part and the imaginary part of the following two complex numbers in terms of rational numbers and using the process of taking the square roots of positive numbers.

$(a) \qquad \left(\frac{2+i}{3-2i}\right)^2.$

$(b) \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}.$

Problem 2 (from Ahlfors p.6, #1). Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

Problem 3 (from Ahlfors p.11, #1). Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

Problem 4 (from Ahlfors p.11, #3). If $|a_i| < 1$ and $\lambda_i \geq 0$ for $i = 1, \cdots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

Problem 5 (from Ahlfors p.15, #2). Prove that the points a_1, a_2, a_3 of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1 a_2 + a_2 a_3 + a_3 a_1$.

Problem 6 (from Ahlfors p.16, #1). (a) Express $\cos 3\varphi$, $\cos 4\varphi$, and $\sin 5\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$ and (b) express $\tan 7\varphi$ in terms of $\tan\varphi$.

Problem 7 (from Ahlfors p.16, #2). Simplify $1+\cos\varphi+\cos 2\varphi+\cdots+\cos n\varphi$ and $\sin\varphi+\sin 2\varphi+\cdots+\sin n\varphi$.

Problem 8 (from Ahlfors p.32, #2). If Q is a polynomial with complex coefficients whose roots $\alpha_1,\cdots,\alpha_n$ are all distinct, and if P is a polynomial of degree $< n$ with complex coefficients, show that

$$\frac{P(z)}{Q(z)}=\sum_{k=1}^{n}\frac{P\left(\alpha_k\right)}{Q'\left(\alpha_k\right)\left(z-\alpha_k\right)},$$

where $Q'(z)$ is the derivative of $Q(z)$.

Problem 9 (from Ahlfors p.32, #3). Use the formula in the preceding problem to prove that there exists a unique polynomial P of degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

Problem 10. Use the method of Cauchy's theorem (as presented in class) to establish the following three integration formulae.

(a) $$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\sqrt{2}\,\pi.$$

(b) $$\int_{0}^{2\pi}\frac{\cos^2 3\theta\,d\theta}{5-4\cos 2\theta}=\frac{3\pi}{8}.$$

(c) $$\int_{0}^{\pi}\sin^{2n}\theta\,d\theta=\frac{(2n)!}{2^{2n}\left(n!\right)^2}\,\pi.$$