Homework Assigned on September 28, 2006
due October 5, 2006

Problem 1 (from Ahlfors p.108, #2, #7; p.120, #1, #3). (a) Let $r > 0$ and x be the real part of the complex variable z. Compute

$$\oint_{|z|=r} x\, dz$$

for the positive sense of the circle in two ways: first by use of a parameter, and second, by observing that

$$x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$$

on the circle.

(b) Suppose $P(z)$ is a polynomial of a single complex variable with complex coefficients and C is the circle $|z - a| = R$, where $a \in \mathbb{C}$ and $R > 0$. Show that

$$\int_C P(z) d\bar{z} = -2\pi i R^2 P'(a).$$

(c) Compute

$$\oint_{|z|=1} \frac{e^z}{z} dz$$

(d) Let $\rho > 0$ and $a \in \mathbb{C}$ with $|a| \neq \rho$. Compute

$$\oint_{|z|=\rho} \frac{|dz|}{|z - a|^2}.$$

Hint: make use of the equation $z\bar{z} = \rho^2$ and

$$|dz| = -i\rho \frac{dz}{z}.$$

(e) Verify that

$$\oint_{|z|=4} \frac{z^{15}}{(z^2 + 1)^2 (z^4 + 2)^3} dz = 2\pi i$$

by using the change of variables $z = \frac{1}{w}$.

Problem 2 (from Ahlfors p.108, #5). Suppose γ is a smooth closed curve in $\mathbb{C}$ and D is a domain in $\mathbb{C}$ which contains γ. Suppose $f(z)$ is a holomorphic function on D. Show that

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary.

Hint: Consider the integration of the exact differential $d\left(f(z)\overline{f(z)}\right)$ over the closed curve γ.

Problem 3 (from Ahlfors p.130, #5). Prove that an isolated singularity of $f(z)$ is removable as soon as either $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded above or below. *Hint:* Apply fractional linear transformation

$$w \mapsto \frac{aw+b}{cw+d}$$

(with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$).

Problem 4. Let $n \in \mathbb{N}$. Suppose $f(z)$ is a holomorphic function on all of $\mathbb{C}$. Suppose for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that

$$|f(z)| \leq \varepsilon |z|^{n+1} + A_\varepsilon \text{ for } z \in \mathbb{C}$$

is satisfied. Show that $f(z)$ is a polynomial of degree at most n.

Hint: Use

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+2}} \quad \text{for } |z| < R$$

and

$$\left| \int_C g(z) dz \right| \leq \left(\sup_{z \in C} |g| \right) \cdot (\text{length of } C)$$

to show by letting $R \to \infty$ and then $\varepsilon \to 0$ that $f^{(n+1)}(z) = 0$ for every $z \in \mathbb{C}$.

Problem 5. The coefficient of the n-th power of z in the power series expansion, about $z = 0$, of the function

$$f(z) = \frac{4 - z^2}{4 - 4zt + z^2} \quad (-1 \leq t \leq 1)$$

is called a Tchebychev polynomial (notation: $T_n(t)$). Prove that

$$T_n(t) = \frac{1}{2^{n-1}} \cos\left(n \cos^{-1} t\right).$$