Partial Fraction Expansion of Meromorphic Functions, Infinite Product Expansion of Entire Functions, and Summation of Series by Residues and the Cotangent Function

Partial Fraction Expansion of Meromorphic Functions. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles are simple $\{a_n\}_{1\leq n<\infty}$ with

$$0<|a_1|\leq|a_2|\leq\cdots$$

so that the residue of $f(z)$ at a_n is b_n. Suppose that there is a sequence of closed contours C_n such that C_n includes $a_1,\cdots,a_n$ but no other poles. Assume that the distance R_n from C_n to the origin goes to infinity as $n\to\infty$ and the length L_n of C_n is of the order $O(R_n)$. Assume that on C_n we have $f(z)=o(R_n^{p+1})$, where $o(\cdot)$ is the Landau symbol which in this case means that

$$(\natural)\qquad \lim_{n\to\infty}\frac{\sup_{z\in C_n}|f(z)|}{R_n^{p+1}}=0.$$

We are going to apply the theorem of residue to the integral

$$(*)\qquad \frac{1}{2\pi i}\int_{C_n}\frac{f(w)}{w^{p+1}(w-z)}dw.$$

The reason for this expression is as follows. To explicitly write down the value of $f(z)$ we should like to use Cauchy's integral formula

$$(\dagger)\qquad f(z)=\frac{1}{2\pi i}\int_{C_n}\frac{f(w)}{w-z}dw$$

over C_n *if the function* $f(z)$ *were holomorphic on the domain enclosed by* C_n, otherwise we have to add the contributions from the poles of $f(z)$ inside C_n. In either case, in general we have no way of explicitly computing the right-hand of ($\dagger$). If somehow after modifying the integral we are able to make the integral over C_n go to zero or at least go to some explicitly computable expression as $n\to 0$, we would then have a way of explicitly writing down $f(w)$.

Because of the condition ($\natural$), if we put in the factor w^p in the denominator of the integrand of the right-hand side of ($\dagger$), the integral over C_n would go to zero as $n\to 0$. That is the reason why we use the integral in ($*$). We do have to pay a price for putting in the additional factor w^p in the denominator of the integrand of the right-hand side of ($\dagger$). The price is the new residue of the integrand at the point $w=0$, which we are going to compute.

The residue at $w = 0$ is obtained by expanding

$$\frac{f(w)}{w^p(w-z)}$$

in Laurent series in w around $w = 0$. We do it separately for

$$f(w), \quad \frac{1}{w^p}, \quad \text{and} \quad \frac{1}{w-z}$$

and then take their products. So we have

$$\frac{f(w)}{w^{p+1}(w-z)} = \frac{-1}{w^{p+1}}\left(\frac{1}{z} + \frac{w}{z^2} + \frac{w^2}{z^3} + \cdots\right)\left(f(0) + f'(0)w + \frac{1}{2}f''(0)w^2 + \cdots\right)$$

and the coefficient of $\frac{1}{w}$ is

$$-\frac{1}{z}\left(\frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!}\right).$$

The residue at $w = z$ is given by

$$\frac{f(z)}{z^{p+1}}.$$

The residue at a_n is

$$\frac{b_n}{a_n^{p+1}(a_n - z)}.$$

Since as $n \to \infty$ the integral becomes zero, we get

$$-\frac{1}{z}\left(\frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!}\right) + \frac{f(z)}{z^{p+1}} + \sum_{n=1}^{\infty}\frac{b_n}{a_n^{p+1}(a_n - z)} = 0$$

which means that

$$f(z) = f(0) + z\,f'(0) + \cdots + \frac{z^p}{p!}f^{(p)}(0) + \sum_{n=1}^{\infty}\frac{b_n z^{p+1}}{a_n^{p+1}(a_n - z)}$$

$$= \sum_{\nu=0}^{p}\frac{z^\nu}{\nu!}f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z-a_n} + \frac{1}{a_n} + \frac{z^2}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}}\right).$$

The last expression comes from writing z^{p+1} as $(z^{p+1} - a_n^{p+1}) + a_n^{p+1}$ and then factoring

$$z^{p+1} - a_n^{p+1} = (z - a_n) \sum_{\nu=0}^{p} z^\nu a_n^{p-\nu}.$$

Thus, our final conclusion about partial fraction expansion of such a meromorphic function is the following. If $f(z)$ is a meromorphic function on $\mathbb{C}$ whose poles are simple $\{a_n\}_{1\le n<\infty}$ with

$$0 < |a_1| \le |a_2| \le \cdots$$

so that the residue of $f(z)$ at a_n is b_n, then

$$f(z) = \sum_{\nu=0}^{p} \frac{z^\nu}{\nu!} f^{(\nu)}(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} + \frac{z^2}{a_n^2} + \cdots + \frac{z^p}{a_n^{p+1}} \right).$$

Example. $f(z) = \operatorname{cosec} z - \frac{1}{z}$ and C_n is the square with corners at the four points

$$\pi \left(n + \frac{1}{2} \right) (\pm 1 \pm i).$$

Observe that when $|y| > \frac{\pi}{2}$ we have

$$|\operatorname{cosec} z| \le 2(e^{\pi/2} - e^{-\pi/2})^{-1}$$

and hence uniformly bounded. Also observe that $\operatorname{cosec} z$ is bounded on the line joining $\frac{1}{2}(1-i)\pi$ to $\frac{1}{2}(1+i)\pi$ and we can use periodicity and conclude that $\operatorname{cosec} z$ is uniformly bounded on C_n. Hence

$$\operatorname{cosec} z - \frac{1}{z} = {\sum}'^{\infty}_{n=-\infty} (-1)^n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

because the residue of $\operatorname{cosec} z$ at $n\pi$ is $(-1)^n$.

Example. $f(z) = \cot z - \frac{1}{z}$ and C_n is the square with corners at the four points

$$\pi \left(n + \frac{1}{2} \right) (\pm 1 \pm i).$$

Observe that when $|y| > \frac{\pi}{2}$ we have

$$|\cot z| \le \left| \frac{e^{2iz} + 1}{e^{2iz} - 1} \right| \le \frac{e^{2y} + 1}{e^{2y} - 1} = 1 + \frac{2}{e^{2y} - 1} \le 1 + \frac{2}{e^{\pi} - 1}$$

and hence uniformly bounded. Also observe that $\cot z$ is bounded on the line joining $\frac{1}{2}(1-i)\pi$ to $\frac{1}{2}(1+i)\pi$ and we can use periodicity and conclude that $\operatorname{cosec} z$ is uniformly bounded on C_n. Hence

$$\cot z - \frac{1}{z} = \sum\nolimits_{n=-\infty}^{\prime\infty}\left(\frac{1}{z-n\pi}+\frac{1}{n\pi}\right),$$

because the residue of $\operatorname{cosec} z$ at $n\pi$ is 1. Thus

$$\pi\,\cot\pi z = \frac{1}{z} + \sum\nolimits_{n=-\infty}^{\prime\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

(Here $\sum'$ means that the index value of $n=0$ is excluded from the summation.)

Infinite Product Expansion of Entire Functions. We can apply the partial fraction argument to

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z)$$

and then afterwards integrate from 0 to z and then take the exponential and we get, for example, for the case $p=0$

$$f(z) = f(0)\exp\left(z\frac{f'(0)}{f(0)}\right)\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{a_n}\right)\exp\frac{z}{a_n}\right].$$

Example. $f(z)=\sin\pi z$. We have

$$\frac{d}{dz}\left(\log\sin\pi z\right) = \pi\cot\pi z = \frac{1}{z} + \sum\nolimits_{n=-\infty}^{\prime\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

and

$$\frac{d}{dz}\left(\log\frac{\sin\pi z}{\pi z}\right) = \pi\cot\pi z = \sum\nolimits_{n=-\infty}^{\prime\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

Since

$$\frac{d}{dz}\log\left(\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right) = \frac{1}{z-n}+\frac{1}{n},$$

when we integrate from $z=0$ to z, we get

$$\int_{\zeta=0}^{\zeta=z}\left(\frac{1}{\zeta-n}+\frac{1}{n}\right)d\zeta = \log\left(\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right).$$

After exponentiating, we obtain

$$\frac{\sin \pi z}{\pi z} = C \prod_{n=-\infty}^{'\infty} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.$$

Setting $z = 1$, we determine C to be 1 and get the following factorization of $\sin \pi z$ as a canonical product.

$$\sin \pi z = \pi z \prod_{n=-\infty}^{'\infty} \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}.$$

(Here $\prod'$ means that the index value of $n = 0$ is excluded from the infinite product.)

Summation of Series by Residues and the Cotangent Function. Suppose $f(z)$ is a rational function whose poles are simple nonintegers $a_1, \cdots, a_k$ with residues $b_1, \cdots, b_k$ such that the degree of the denominator of f is at least two more than that of its numerator. Let C_n be the square with corners at the four points

$$\left(n + \frac{1}{2}\right)(\pm 1 \pm i).$$

The integral

$$\int_{C_n} \pi \cot \pi z \, f(z) dz$$

goes to zero as $n \to \infty$. Hence

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{\nu=1}^{k} b_\nu \cot \pi a_\nu.$$

If we use $\operatorname{cosec} \pi z$ instead of $\cot \pi z$ we can obtain the sum of the series $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$.

Example. To sum the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}$$

for $a > 0$, we use the function

$$f(z) = \frac{1}{z^2 + a^2}$$

which has simple poles at $z = ai$ and at $z = -ai$ with residues respectively $\frac{-i}{2a}$ and $\frac{i}{2a}$. Thus

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\pi \left(\frac{-i \cot \pi a i}{2a} + \frac{i \cot \pi (-ai)}{2a} \right) = \frac{\pi}{a} \coth \pi a.$$