Integration of Rational Functions of Sine and Cosine

The kind of integrals we would like to compute by using

(i) the application of Stokes's theorem to the integrals of rational functions,

(ii) partial fractions of rational functions

is the following.

$$\int_0^{2\pi} R(\sin\theta,\cos\theta)\,d\theta,$$

where $R(\cdot,\cdot)$ is a rational function. Let us start with on example.

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} \quad \text{with } 0<a<1.$$

By using the parametrization $z=e^{i\theta}$, we get

$$d\theta=\frac{dz}{iz}, \quad \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right), \quad \sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

so that the integral becomes

$$\oint_{|z|=1}\frac{dz}{iz\left(1-az-\frac{p}{z}+p^2\right)}=\oint_{|z|=1}\frac{idz}{az^2-(a^2+1)\,z+a}.$$

(Actually we do not need $\sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$ for this example and we include it to show how the general case is handled.) The two roots of the quadratic equation $az^2-(a^2+1)\,z+a=0$ are $z=a$ and $z=\frac{1}{a}$ so that

$$az^2-\left(a^2+1\right)z+a=a\,(z-a)\left(z-\frac{1}{a}\right).$$

The partial fraction of

$$\frac{1}{az^2-(a^2+1)\,z+a}=\frac{A}{z-a}+\frac{B}{z-\frac{1}{a}}$$

with undetermined coefficients A and B can be determined by multiplying both sides of

$$\frac{1}{a\,(z-a)\left(z-\frac{1}{a}\right)}=\frac{A}{z-a}+\frac{B}{z-\frac{1}{a}}$$

by $z-a$ and setting $z=a$ to get $A=\frac{1}{a^2-1}$ and multiplying both sides by $z-\frac{1}{a}$ and setting $z=\frac{1}{a}$ to get $B=\frac{1}{1-a^2}$. Thus the original integral can be written as

$$\oint_{|z|=1}\frac{idz}{az^2-(a^2+1)\,z+a}=\frac{i}{1-a^2}\oint_{|z|=1}\left(\frac{-1}{z-a}+\frac{1}{z-\frac{1}{a}}\right)dz.$$

The second integral is zero, because the function $\frac{1}{z-\frac{1}{a}}$ is continuously differentiable and satisfies the Cauchy-Riemann equation on the closed disk $|z|\leq 1$. The contribution from the function $\frac{-1}{z-a}$ makes the value of the integral

$$\frac{i}{1-a^2}\left(-2\pi i\right)=\frac{2\pi}{1-a^2}$$

which is the final answer. As long as we are able to do partial fractions, we can evaluate the integral of rational functions of the sine and cosine functions over $[0,2\pi]$.

We now do another integral with a slightly more complicated situation of decomposition into partial fractions. The integral is

$$\int_0^{2\pi}\frac{d\theta}{\left(\alpha+\beta\cos\theta\right)^2}\quad\text{for } \alpha>\beta>0.$$

Again we use the same parametrization $z=e^{i\theta}$ and

$$d\theta=\frac{dz}{iz},\quad \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right),\quad \sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$$

to transform the integral to

$$\oint_{|z|=1}\frac{dz}{iz\left(\alpha+\frac{\beta}{2}\left(z+\frac{1}{z}\right)\right)^2}=-i\oint_{|z|=1}\frac{zdz}{\left(\frac{\beta}{2}z^2+\alpha z+\frac{\beta}{2}\right)^2}.$$

(Again we do not need $\sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$ for this example and we include it to show how the general case is handled.) The two roots of the quadratic equation $\frac{\beta}{2}z^2+\alpha z+\frac{\beta}{2}=0$ are

$$a=\frac{1}{\beta}\left(-\alpha+\sqrt{\alpha^2-\beta^2}\right),\qquad b=\frac{1}{\beta}\left(-\alpha-\sqrt{\alpha^2-\beta^2}\right)$$

so that

$$\frac{\beta}{2}z^2+\alpha z+\frac{\beta}{2}=\frac{\beta}{2}\left(z-a\right)\left(z-b\right)$$

and we need to get the partial fraction expansion of

$$\frac{z}{\left(\frac{\beta}{2}z^2+\alpha z+\frac{\beta}{2}\right)^2}=\frac{4z}{\beta^2\left(z-a\right)^2\left(z-b\right)^2}.$$

The partial fraction decomposition is of the form

$$(*)\qquad \frac{4z}{\beta^2\left(z-a\right)^2\left(z-b\right)^2}=\frac{A}{z-a}+\frac{B}{\left(z-a\right)^2}+\frac{C}{z-b}+\frac{D}{\left(z-b\right)^2}.$$

Clearly the absolute value $|b|$ of $b=\frac{1}{\beta}\left(-\alpha-\sqrt{\alpha^2-\beta^2}\right)$ is >1 because $\alpha>\beta>0$. From

$$|a|\,|b|=\frac{1}{\beta^2}\left(\alpha^2-\left(\alpha^2-\beta^2\right)\right)=\frac{\beta^2}{\beta^2}=1$$

it follows that the absolute value $|a|$ of a is <1. For the computation of the integral in question, in the partial fraction decomposition $(*)$ it suffices to determine the value of A. Multiplying both sides of $(*)$ by $(z-a)^2$, we obtain

$$(\dagger)\qquad \frac{4z}{\beta^2\left(z-b\right)^2}=A(z-a)+B+\frac{C(z-a)^2}{z-b}+\frac{D(z-a)^2}{\left(z-b\right)^2}.$$

Differentiating both sides of $(\dagger)$ and setting $z=a$ yields

$$\begin{aligned}A&=\left(\frac{d}{dz}\,\frac{4z}{\beta^2\left(z-b\right)^2}\right)_{z=a}\\&=\frac{4}{\beta^2\left(a-b\right)^2}-\frac{8a}{\beta^2\left(a-b\right)^3}\\&=\frac{4(a-b)-8a}{\beta^2\left(a-b\right)^3}=\frac{-4(a+b)}{\beta^2\left(a-b\right)^3}.\end{aligned}$$

Since $a+b=\frac{-2\alpha}{\beta}$ and $a-b=\frac{2\sqrt{\alpha^2-\beta^2}}{\beta}$, it follows that

$$A=\frac{\alpha}{\left(\alpha^2-\beta^2\right)^{\frac{3}{2}}}$$

and the integral in question is equal to

$$-i\,A\,2\pi i = \frac{2\pi\alpha}{(\alpha^2-\beta^2)^{\frac{3}{2}}}.$$

Partial Fraction Decomposition. Let $\varphi(z)$, $\psi(z) \in \mathbb{C}[z]$ be without any common factors of degrees respectively m and n. Then any $f(z) \in \mathbb{C}[z]$ of degree $< m+n$ can be uniquely written as $f(z) = A(z)\varphi(z) + B(z)\psi(z)$ with $A(z)$, $B(z) \in \mathbb{C}[z]$ of degrees respectively $< n$ and $< m$. Thus we can write

$$\frac{f(z)}{\varphi(z)\psi(z)} = \frac{A(z)}{\psi(z)} + \frac{B(z)}{\varphi(z)}.$$

Proof. The greatest common divisor of $\varphi(z)$ and $\psi(z)$ is 1. We can thus write

$$1 = \sigma(z)\varphi(z) + \tau(z)\psi(z)$$

for $\sigma(z)$, $\tau(z) \in \mathbb{C}[z]$ some and

$$f(z) = f(z)\sigma(z)\varphi(z) + f(z)\tau(z)\psi(z).$$

Euclidean division yields

$$f(z)\sigma(z) = g(z)\psi(z) + A(z)$$

with the degree of $A(z) < n$. We have

$$\begin{aligned} f(z) &= \left(g(z)\psi(z) + A(z)\right)\varphi(z) + f(z)\tau(z)\psi(z) \\ &= A(z)\varphi(z) + \left(f(z)\tau(z) - g(z)\psi(z)\right)\psi(z). \end{aligned}$$

Since the degree of $f(z) - A(z)\varphi(z)$ is $< m+n$, it follows from

$$f(z) - A(z)\varphi(z) = \left(f(z)\tau(z) - g(z)\psi(z)\right)\psi(z)$$

that the degree of $f(z)\tau(z) - g(z)\psi(z)$ is $< m$ and we can set $B(z) = f(z)\tau(z) - g(z)\psi(z)$ so that $f(z) = A(z)\varphi(z) + B(z)\psi(z)$ with $A(z)$, $B(z) \in \mathbb{C}[z]$ of degrees respectively $< n$ and $< m$.

Suppose $a_1, \cdots, a_\ell \in \mathbb{C}$ are all distinct and $g(z) = \prod_{j=1}^{\ell} (z - a_j)^{k_j}$ and the degree of $f(z) \in \mathbb{C}[z]$ is less than $n := \sum_{j=1}^{\ell} k_j$. Then

$$\frac{f(z)}{g(z)} = \sum_{j=1}^{\ell} \left(\sum_{\nu=1}^{k_j} \frac{A_{j,\nu}}{(z - a_j)^\nu} \right)$$

for some $A_{j,\nu} \in \mathbb{C}$ uniquely. Since $a_1, \cdots, a_\ell \in \mathbb{C}$, by induction on ℓ and using $\varphi(z) = (z - a_1)^{k_1}$ and $\psi(z) = \prod_{j=2}^{\ell} (z - a_j)^{k_j}$, we can write

$$\frac{f(z)}{g(z)} = \sum_{j=1}^{\ell} \frac{h_j(z)}{(z - a_j)^{k_j}}$$

for some $h_j(z) \in \mathbb{C}[z]$ of degree $< k_j$. Finally we can regard $h_j(z)$ as a polynomial in $z - a_j$ with coefficients in $\mathbb{C}$ and get

$$h_j(z) = \sum_{\nu=0}^{q_j} A_{j,\nu+1} (z - a_j)^\nu$$

for some $q_j < k_j$. We need only set $A_{j,\nu+1} = 0$ for $q_j < \nu \leq k_j - 1$ to obtain

$$\frac{f(z)}{g(z)} = \sum_{j=1}^{\ell} \left(\sum_{\nu=1}^{k_j} \frac{A_{j,\nu}}{(z - a_j)^\nu} \right).$$

Fix $1 \leq j \leq \ell$. For uniqueness of A_{j,k_j} we can multiply both sides by $(z - a_j)^{k_j}$ and set $z = a_j$. By considering

$$\frac{f(z)}{g(z)} - \frac{A_{j,k_j}}{(z - a_j)^{k_j}}$$

and descending induction on k_j, we get the uniqueness of $A_{j,\nu}$ for $1 \leq \nu \leq k_j$.

For the evaluation of a general $I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$ we use

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} R\left(\frac{1}{2}\left(z + \frac{1}{z} \right), \frac{1}{2i}\left(z - \frac{1}{z} \right) \right) \frac{dz}{iz}$$

and the partial fraction decomposition

$$\frac{1}{iz}R\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)=\sum_{j=1}^{\ell}\left(\sum_{\nu=1}^{k_j}\frac{A_{j,\nu}}{\left(z-a_j\right)^{\nu}}\right)$$

(with $A_{j,k_j}\neq 0$ and all $a_1,\cdots,a_\ell$ distinct) and get $I=2\pi i\sum_{j\in J}A_{j,1}$ where J consists of all $1\leq j\leq \ell$ with $|a_j|<1$. Note that the case $|a_j|=1$ cannot occur if the integrand of the denominator I is nonzero for any $0\leq\theta\leq 2\pi$. The computation of $A_{j,1}$ is given by

$$\frac{1}{(k_j-1)!}\left(\frac{d^{k_1-1}}{dz^{k_j-1}}\left(\left(z-a_j\right)^{k_j}\frac{1}{iz}R\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\right)\right)_{z=a_j}.$$