Legendre Functions and the Laplace Equation in Spherical Coordinates

Dirichlet Problem on the Unit 3-Dimensional Unit Ball. We are going to introduce the (associated) Legendre functions by considering the solution of the Dirichlet problem of finding a harmonic function u on the unit ball in $\mathbb{R}^3$ with a prescribed boundary value u_1. In other words, we seek to solve the equation

$$\Delta u = 0 \quad \text{on} \quad B$$

with $u = u_1$ on the boundary of B, where Δ is the Laplacian and B is the unit ball in $\mathbb{R}^3$ and u_1 is a given function defined on the boundary of B.

Reduction of Laplace Equation in Spherical Coordinates for Special Product Functions to Three Equations by the Method of the Separation of Variables. Consider the spherical coordinate (r, θ, φ), where r is the distance to the origin, θ is the longitude, and φ is the colatitude. In spherical coordinates (r, θ, φ) the Laplace equation $\Delta u = 0$ for a function $u(r, \theta, \varphi)$ reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2} = 0.$$

Consider the special case where u is a product $R(r)\Phi(\varphi)\Theta(\theta)$. We are going to use the method of separation of variables to reduce the Laplace equation for the special product function $u(r, \theta, \varphi) = R(r)\Phi(\varphi)\Theta(\theta)$ into three ordinary differential equations. The Laplace equation

$$\frac{\Phi\Theta}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Theta}{r^2\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right) + \frac{R\Phi}{r^2\sin^2\varphi}\frac{d^2\Theta}{d\theta^2} = 0$$

for the special product function can be rewritten as

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Phi\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right) - \frac{1}{\Theta\sin^2\varphi}\frac{d^2\Theta}{d\theta^2}.$$

Both sides must be equal to a constant which we denote by λ. We now have the reduction into the following two differential equations.

$$(\sharp) \qquad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda,$$

$$-\frac{1}{\Phi\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)-\frac{1}{\Theta\sin^2\varphi}\frac{d^2\Theta}{d\theta^2}=\lambda.$$

We first observe that λ must be nonnegative, because the second differential equation can be rewritten as

$$-\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\left(\Theta\Phi\right)}{d\varphi}\right)-\frac{1}{\sin^2\varphi}\frac{d^2\left(\Theta\Phi\right)}{d\theta^2}=\lambda\left(\Theta\Phi\right).$$

Multiplying the equation by $\Theta\Phi\sin\varphi$ and integrating over $0\leq\theta\leq 2\pi$ and $0\leq\varphi\leq\pi$ to get

$$\int_{\theta=0}^{\infty}\int_{\varphi=0}^{\pi}\left(\sin\varphi+\frac{1}{\sin\varphi}\right)\left(\frac{d\left(\Theta\Phi\right)}{d\varphi}\right)^2 d\theta\,d\varphi=\lambda\int_{\theta=0}^{\infty}\int_{\varphi=0}^{\pi}\sin\varphi\left(\frac{d\left(\Theta\Phi\right)}{d\varphi}\right)^2 d\theta\,d\varphi$$

due to the vanishing of $\sin\varphi$ at $\varphi=0$ and $\varphi=\pi$, we conclude the constant λ must be nonnegetive, because $\sin\varphi>0$ for $0<\varphi<\pi$.

We now reduce the differential equation

$$-\frac{1}{\Phi\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)-\frac{1}{\Theta\sin^2\varphi}\frac{d^2\Theta}{d\theta^2}=\lambda$$

further by separation of variables by first rewriting it as

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-\frac{\sin\varphi}{\Phi}\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)-\lambda\sin^2\varphi.$$

Both sides must be equal to a constant which we denote by μ. We now have its reduction into the following two differential equations.

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=\mu,$$

$$\sin\varphi\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)+\left(\lambda\sin^2\varphi+\mu\right)\Phi=0.$$

Since the function $\Theta=\Theta\left(\theta\right)$ in the first differential equation

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=\mu \tag{\&}$$

is periodic in θ of period 2π, it follows that μ must be nonpositive and equal to $-m^2$ for some nonnegative integer m. The second differential equation now becomes

$$\sin\varphi\frac{d}{d\varphi}\left(\sin\varphi\frac{d\Phi}{d\varphi}\right)+\left(\lambda\sin^2\varphi-m^2\right)\Phi=0.$$

We now use the substitution $\xi = \cos\varphi$ to get rid of the sine and cosine functions in this differential equation. In terms of the new independent variable ξ the differential equation becomes

$$\left(1-\xi^2\right)\frac{d^2\Phi}{d\xi^2} - 2\xi\frac{d\Phi}{d\xi} + \left(\lambda - \frac{m^2}{1-\xi^2}\right)\Phi = 0.$$

Change the symbol Φ to y and the symbol ξ to x. We get the differential equation

$$\left(1-x^2\right)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0.$$

Differential Equations for Legendre Functions and Removal of the Constant m by Transformation of Dependent Variable. We are now going to apply a transformation to the dependent variable to get rid of the constant m in the differential equation

$$(\natural) \qquad \left(1-x^2\right)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0$$

so that with a new dependent variable y the differential equation becomes

$$(*) \qquad \left(1-x^2\right)y'' - 2xy' + \lambda\, y = 0.$$

We will do the transformation of the dependent variable in the reverse direction by starting out with the differential equation $(*)$ and apply transformations to get to the differential equation $(\natural)$. We differentiate m times the differential equation $(*)$ and use the formula for higher-derivatives of a product of two functions

$$\frac{d^m}{dx^m}(fg) = \sum_{k=0}^{m}\binom{m}{k}\frac{d^k f}{dx^k}\,\frac{d^{m-k}g}{dx^{m-k}}$$

to get

$$\left(1-x^2\right)\frac{d^{m+2}y}{dx^{m+2}} + m\,(-2x)\,\frac{d^{m+1}y}{dx^{m+1}} + \frac{m(m-1)}{2}\,(-2)\,\frac{d^m y}{dx^m}$$
$$-2x\frac{d^{m+1}y}{dx^{m+1}} - 2m\frac{d^m y}{dx^m} + \lambda\frac{d^m y}{dx^m} = 0.$$

We introduce the variable

$$v = \frac{d^m y}{dx^m}$$

to transform the last differential equation to

$$(\dagger) \qquad \left(1-x^2\right)v'' - 2(m+1)xv' + (\lambda - m(m+1))\,v = 0.$$

Now we let $w = (1-x^2)^{\frac{m}{2}}\,v$ and compute its derivatives of first and second order

$$\begin{aligned} w' &= \frac{m}{2}\left(1-x^2\right)^{\frac{m}{2}-1}(-2x)\,v + \left(1-x^2\right)^{\frac{m}{2}}v', \\ w'' &= \frac{m}{2}\left(\frac{m}{2}-1\right)\left(1-x^2\right)^{\frac{m}{2}-2}(-2x)^2\,v - m\left(1-x^2\right)^{\frac{m}{2}-1}v \\ &\quad + m\left(1-x^2\right)^{\frac{m}{2}-1}(-2x)\,v' + \left(1-x^2\right)^{\frac{m}{2}}v''. \end{aligned}$$

We add up the following three equations

$$\begin{aligned} \left(1-x^2\right)w'' &= \frac{m}{2}\left(\frac{m}{2}-1\right)\left(1-x^2\right)^{\frac{m}{2}-1}(-2x)^2\,v - m\left(1-x^2\right)^{\frac{m}{2}}v \\ &\quad + m\left(1-x^2\right)^{\frac{m}{2}}(-2x)\,v' + \left(1-x^2\right)^{\frac{m}{2}+1}v''. \\ -2xw' &= \frac{m}{2}\left(1-x^2\right)^{\frac{m}{2}-1}(-2x)^2\,v + \left(1-x^2\right)^{\frac{m}{2}}(-2x)\,v'. \\ \left(\lambda - \frac{m^2}{1-x^2}\right)w &= \lambda\left(1-x^2\right)^{\frac{m}{2}}v - m^2\left(1-x^2\right)^{\frac{m}{2}-1}v \end{aligned}$$

to get

$$\begin{aligned} &\left(1-x^2\right)w'' - 2xw' + \left(\lambda - \frac{m^2}{1-x^2}\right)w \\ &\quad = \left(1-x^2\right)^{\frac{m}{2}}\left(\left(1-x^2\right)v'' - 2(m+1)xv' + Av\right), \end{aligned}$$

where

$$\begin{aligned} A &= m(m-2)\frac{x^2}{1-x^2} - m + 2m\frac{x^2}{1-x^2} + \lambda - \frac{m^2}{1-x^2} \\ &= m^2\frac{x^2}{1-x^2} - m + \lambda - \frac{m^2}{1-x^2} \\ &= -m^2 - m + \lambda = \lambda - m(m+1). \end{aligned}$$

From $(\dagger)$ it follows that

$$\left(1-x^2\right)w'' - 2xw' + \left(\lambda - \frac{m^2}{1-x^2}\right)w = 0.$$

which is the same as $(*)$ when we change the symbol w to y. Thus the transformation

$$(\ddagger) \qquad y \mapsto \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m y}{dx^m}.$$

of the dependent variable y changes the differential equation $(\natural)$ to the differential equation $(*)$. The differential equation $(*)$ is the differential equation for Legendre functions.

Solution of Legendre's Differential Equation by Power Series With Undetermined Coefficients and the Determination of the Constant λ. We use the power series

$$y(x)=\sum_{k=0}^{\infty} a_k x^k$$

with undetermined coefficients to solve the Legendre differential equation

$$\left(1-x^2\right) y'' - 2xy' + \lambda\, y = 0.$$

We get

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^k - 2\sum_{k=0}^{\infty} k a_k x^k + \lambda \sum_{k=0}^{\infty} a_k x^k = 0.$$

After we change the dummy variable k in the first sum to $k+2$, we get

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - k(k+1)a_k + \lambda a_k\right) x^k = 0.$$

Thus we have the following recurrent relation

$$a_{k+2} = \frac{k(k+1)-\lambda}{(k+2)(k+1)}\, a_k \quad \text{for} \quad k \geq 0.$$

The logic is that if a power series satisfies the differential equation, then its coefficients must satisfy the above recurrent relation. By the ratio test we conclude that the radius of convergence of the power series is at least 1. Note that we are free to choose values for a_0 and a_1. We now prove the following claim.

Claim. If there exists some k_0 such that a_{k_0} is nonzero and $k_0\,(k_0+1)>\lambda$, then the power series solution $y(x)=\sum_{k=0}^{\infty}a_kx^k$ of Legendre's differential equation $(*)$ cannot converge both at $x=1$ and at $x=-1$.

To prove the Claim, we assume the contrary and proceed to derive a contradiction. Since $k_0\,(k_0+1)>\lambda$ and $a_{k_0}\neq 0$, it follows that $a_{k_0+2\ell}$ and a_{k_0} are all of the same sign for $\ell\geq 0$. Choose η equal to 1 or -1 such that $\eta a_{k_0+2\ell}>0$ for all $\ell\geq 0$. From the recurrent relation which we rewrite as

$$\begin{aligned} a_{k+2} &= \frac{k(k+1)}{(k+2)(k+1)}\,a_k-\frac{\lambda}{(k+2)(k+1)}\,a_k \\ &= \frac{k}{k+2}\,a_k-\frac{\lambda}{(k+2)(k+1)}\,a_k. \end{aligned}$$

For $\ell\geq 1$ we will by induction on $\ell\geq 1$ verify

$$\begin{aligned} \eta\,a_{k_0+2\ell} \geq{} & \frac{k_0\,\eta\,a_{k_0}}{k_0+2\ell}-\frac{\lambda\,\eta\,a_{k_0}}{(k_0+2\ell)\,(k_0+1)} \\ & -\sum_{j=0}^{\ell-2}\frac{\lambda\,k_0\,\eta\,a_{k_0}}{(k_0+2j+2)\,(k_0+2j+3)\,(k_0+2j+4)}. \end{aligned}$$

The cases of $\ell=1$ is just the recurrent relation. The case $\ell=2$ comes from

$$\begin{aligned} a_{k_0+4} &= \frac{k_0+2}{k_0+4}\,a_{k_0+2}-\frac{\lambda}{(k_0+4)(k_0+3)}\,a_{k_0+2} \\ &= \frac{k_0+2}{k_0+4}\left(\frac{k_0}{k_0+2}\,a_{k_0}-\frac{\lambda}{(k_0+2)(k_0+1)}\,a_{k_0}\right) \\ &\quad -\frac{\lambda}{(k_0+4)(k_0+3)}\left(\frac{k_0}{k_0+2}\,a_{k_0}-\frac{\lambda}{(k_0+2)(k_0+1)}\,a_{k_0}\right) \\ &\geq \frac{k_0}{k_0+4}\,a_{k_0}-\frac{\lambda}{(k_0+4)(k_0+1)}\,a_{k_0}-\frac{\lambda k_0}{(k_0+4)(k_0+3)(k_0+2)}\,a_k. \end{aligned}$$

Suppose it has been verified for $\ell\geq 2$ and we verify the situation where ℓ is replaced by $\ell+1$ and get

$$\begin{aligned} \eta\,a_{k_0+2\ell+2} &= \frac{k_0+2\ell}{k_0+2\ell+2}\,\eta\,a_{k_0+2\ell}-\frac{\lambda}{(k_0+2\ell+2)(k_0+2\ell+1)}\,\eta\,a_{k_0+2\ell} \\ &\geq \frac{k_0+2\ell}{k_0+2\ell+2}\left(\frac{k_0\,\eta\,a_{k_0}}{k_0+2\ell}-\frac{\lambda\,\eta\,a_{k_0}}{(k_0+2\ell)\,(k_0+1)}\right. \end{aligned}$$

$$
\begin{aligned}
&\left.-\sum_{j=0}^{\ell-2}\frac{\lambda\, k_0\,\eta\, a_{k_0}}{(k_0+2j+2)\,(k_0+2j+3)\,(k_0+2j+4)}\right) \\
&-\frac{\lambda}{(k_0+2\ell+2)(k_0+2\ell+1)}\left(\frac{k_0\,\eta\, a_{k_0}}{k_0+2\ell}-\frac{\lambda\,\eta\, a_{k_0}}{(k_0+2\ell)\,(k_0+1)}\right. \\
&\qquad\left.-\sum_{j=0}^{\ell-2}\frac{\lambda\, k_0\,\eta\, a_{k_0}}{(k_0+2j+2)\,(k_0+2j+3)\,(k_0+2j+4)}\right) \\
&\geq\frac{k_0\,\eta\, a_{k_0}}{k_0+2\ell+2}-\frac{\lambda\,\eta\, a_{k_0}}{(k_0+2\ell+2)\,(k_0+1)} \\
&\qquad\qquad-\sum_{j=0}^{\ell-1}\frac{\lambda\, k_0\,\eta\, a_{k_0}}{(k_0+2j+2)\,(k_0+2j+3)\,(k_0+2j+4)},
\end{aligned}
$$

because $(k_0+2\ell)\,(k_0+2\ell+1)-\lambda>0$. Summing up over all positive integers ℓ, we get

$$
\begin{aligned}
\sum_{\ell=1}^{\infty}\eta\, a_{k_0+2\ell+2}\geq&\sum_{\ell=1}^{\infty}\frac{k_0\,(k_0+1)-\lambda}{k_0+2\ell+2}\,\eta\, a_{k_0} \\
&-\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1}\frac{\lambda\, k_0\,\eta\, a_{k_0}}{(k_0+2j+2)\,(k_0+2j+3)\,(k_0+2j+4)}.
\end{aligned}
$$

Since both $k_0\,(k_0+1)-\lambda$ and $\eta\, a_{k_0}$ are positive, it follows that

$$
\sum_{\ell=1}^{\infty}\frac{k_0\,(k_0+1)-\lambda}{k_0+2\ell+2}\,\eta\, a_{k_0}
$$

diverges to $+\infty$. Since the term

$$
\sum_{\ell=1}^{\infty}\sum_{j=0}^{\ell-1}\frac{\lambda\, k_0\,\eta\, a_{k_0}}{(k_0+2j+2)\,(k_0+2j+3)\,(k_0+2j+4)}
$$

is finite, it follows that $\sum_{\ell=1}^{\infty}\eta\, a_{k_0+2\ell+2}$ diverges to $+\infty$. This contradicts the assumption of the convergence of

$$
y(x)=\sum_{k=0}^{\infty}a_k x^k
$$

at both $x = 1$ and $x = -1$, because its convergence at both $x = 1$ and $x = -1$ would imply that both

$$\sum_{\ell=0}^{\infty} a_{2\ell}$$

and

$$\sum_{\ell=0}^{\infty} a_{2\ell+1}$$

converge. This finishes the proof of the Claim.

From the Claim it follows that if the power series solution of $(*)$ is finite at $x = 1$ and $x = -1$, then there cannot exist any k_0 with $a_{k_0} \neq 0$ and $k_0(k_0+1) > \lambda$. This can happen only if $\lambda = n(n+1)$ for some integer $n \geq 0$. When $\lambda = n(n+1)$, we can choose a_0 and a_1 in the following way. If n is even, we let $m = \frac{n}{2}$ and set

$$a_0 = \frac{(-1)^m}{2^m} \frac{(2m)!}{m!}$$

and $a_1 = 0$. If n is odd, we let $m = \frac{n}{2} - 1$ and set $a_0 = 0$ and

$$a_1 = \frac{(-1)^m}{2^m} \frac{(2m+2)!}{(m+1)!}.$$

This rather involved way of choosing a_0 and a_1 is to get the finite power series (*i.e.*, polynomial) has the unified expression

$$\frac{1}{2^n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{(2n-2k)!}{(n-2k)!(n-k)!} x^{n-2k},$$

which is called *Legendre's polynomial of degree n* and is denoted by $P_n(x)$.

Now that we have the value of $\lambda = n(n+1)$ we can go back to the equation $(\sharp)$ and come up with one solution $R(r) = r^n$ which satisfies the equation $(\sharp)$ when $\lambda = n(n+1)$.

Rodrigues' Formula. Legendre's polynomial

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{m} \frac{(-1)^k}{k!} \frac{(2n-2k)!}{(n-2k)!(n-k)!} x^{n-2k},$$

of degree n can be put into a simpler form involved differentiation known as *Rodrigues' formula* in the following way. Again let $n = 2m$ if n is even and let $n = 2m + 1$ if n is odd. Since

$$\frac{d^n}{dx^n}\, x^{2n-2k} = \frac{(2n-2k)!}{(n-2k)!}\, x^{n-2k},$$

it follows that

$$P_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n} \sum_{k=0}^{m} (-1)^k \,\frac{(n!}{k!(n-k)!}\, x^{2n-2k},$$

which we can rewrite as

$$(\flat) \qquad P_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}\left(x^2-1\right)^{2n-2k}.$$

The formula ($\flat$) is *Rodrigues' formula.*

Motivated by the above discussion of the removal of m in the differential equation ($\natural$) by the change of the dependent variable ($\ddagger$) to transform it to Legendre's differential equation ($*$), we define the *associated Legendre function*

$$P_n^m(x) = \frac{\left(1-x^2\right)^{\frac{m}{2}}}{2^n n!}\,\frac{d^{n+m}\left(x^2-1\right)^n}{dx^{n+m}}$$

which satisfies the differential equation ($\natural$)

Orthogonality and Norms of Associated Legendre Functions. For problems involving Laplacian in spherical coordinates we need to consider for a fixed integer $m \geq 0$ the expansion of a function on $[-1, 1]$ in terms of the associated Legendre functions P_n^m as n varies in the set of all nonnegative integers. For that purpose we have to use the orthogonality of the family $\{P_n^m\}_{0 \leq n < \infty}$ for a fixed integer $m \geq 0$. We now verify that for nonnegative integers m, p, n with $n > p$ the two functions P_p^m and P_n^m are orthogonal as functions over $[-1, 1]$ from the second-order differential equations ($\ddagger$) which they respectively satisfy.

$$(\natural)_n \qquad \left(1-x^2\right)\left(P_n^m\right)'' - 2x\left(P_n^m\right)' + \left(n(n+1) - \frac{m^2}{1-x^2}\right) P_n^m = 0,$$

$$(\natural)_p \qquad \left(1-x^2\right)\left(P_p^m\right)'' - 2x\left(P_p^m\right)' + \left(p(p+1) - \frac{m^2}{1-x^2}\right) P_p^m = 0.$$

We multiply $(\natural)_n$ by P_p^m and multiply $(\natural)_p$ by P_n^m and take their differences to get

$$\left(\left(1-x^2\right)\left(P_p^m\left(P_n^m\right)'-P_n^m\left(P_p^m\right)'\right)\right)'+(n-p)(n-p+1)P_p^m\,P_n^m=0.$$

By integrating over $[-1,1]$ with respect to x, we obtain

$$\int_{x=-1}^{1}P_p^m(x)\,P_n^m(x)\,dx=0\quad\text{for}\quad n>p.$$

We now compute the L^2 norm of $P_n^m(x)$ over $[-1,1]$ by first computing the case of $m=1$ by integration by parts and then the general case by induction on m and by using differential equation $(\natural)_n$.

$$\int_{x=-1}^{1}\left(P_n(x)\right)^2\,dx=\int_{x=-1}^{1}\left(\frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}\right)\left(\frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}\right)dx$$
$$=(-1)^n\int_{x=-1}^{1}\frac{1}{2^{2n}\left(n!\right)^2}\left(x^2-1\right)^n\frac{d^{2n}\left(x^2-1\right)^n}{dx^{2n}}dx$$
$$=(-1)^n\int_{x=-1}^{1}\frac{(2n)!}{2^{2n}\left(n!\right)^2}\left(x^2-1\right)^n\,dx$$
$$=(-1)^n\int_{x=-1}^{1}\frac{(2n)!}{2^{2n}\left(n!\right)^2}\left(x+1\right)^n\left(x-1\right)^n\,dx$$
$$=(-1)^n\int_{x=-1}^{1}\frac{(2n)!}{2^{2n}\left(n!\right)^2}\frac{-n}{n+1}\left(x+1\right)^{n+1}\left(x-1\right)^{n-1}\,dx$$
$$=(-1)^n\int_{x=-1}^{1}\frac{(2n)!}{2^{2n}\left(n!\right)^2}\frac{-n}{n+1}\,\frac{-(n-1)}{n+2}\left(x+1\right)^{n+2}\left(x-1\right)^{n-2}\,dx$$
$$=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$=(-1)^n\int_{x=-1}^{1}\frac{(2n)!}{2^{2n}\left(n!\right)^2}\frac{-n}{n+1}\,\frac{-(n-1)}{n+2}\,\cdots\,\frac{-1}{2n}\left(x+1\right)^{2n}\,dx$$
$$=(-1)^n\frac{(2n)!}{2^{2n}\left(n!\right)^2}\frac{-n}{n+1}\,\frac{-(n-1)}{n+2}\,\cdots\,\frac{-1}{2n}\,\frac{1}{2n+1}\left(x+1\right)^{2n+1}\Big|_{x=-1}^{x=1}$$
$$=(-1)^n\frac{(2n)!}{2^{2n}\left(n!\right)^2}\frac{-n}{n+1}\,\frac{-(n-1)}{n+2}\,\cdots\,\frac{-1}{2n}\,\frac{1}{2n+1}\,2^{2n+1}=\frac{2}{2n+1}.$$

We conclude that the L^2 norm of $P_n(x)$ over $[-1,1]$ is $\sqrt{\frac{2}{2n+1}}$. We now use induction on the nonnegative integer m to compute the L^2 norm of $P_n^m(x)$ over $[-1,1]$. Since

$$\frac{d}{dx}P_n^m(x)=\frac{d}{dx}\left(\frac{\left(1-x^2\right)^{\frac{m}{2}}}{2^n n!}\frac{d^{n+m}\left(x^2-1\right)^n}{dx^{n+m}}\right)$$

$$=\frac{m}{2}\frac{\left(1-x^2\right)^{\frac{m-2}{2}}}{2^n n!}(-2x)\frac{d^{n+m}\left(x^2-1\right)^n}{dx^{n+m}}+\frac{\left(1-x^2\right)^{\frac{m}{2}}}{2^n n!}\frac{d^{n+m+1}\left(x^2-1\right)^n}{dx^{n+m+1}},$$

it follows that

$$\left(1-x^2\right)^{\frac{1}{2}}\frac{d}{dx}P_n^m(x)=-mx\left(1-x^2\right)^{\frac{-1}{2}}P_n^m(x)+P_n^{m+1}(x),$$

which can be rewritten as

$$P_n^{m+1}(x)=\left(1-x^2\right)^{\frac{1}{2}}\frac{d}{dx}P_n^m(x)+mx\left(1-x^2\right)^{\frac{-1}{2}}P_n^m(x).$$

By using this formula we can compute the L^2 norm of $P_n^{m+1}(x)$ over $[-1,1]$ by

$$\int_{x=-1}^{1}\left(P_n^{m+1}(x)\right)^2\,dx$$

$$=\int_{x=-1}^{1}\left(\left(1-x^2\right)\left(\left(P_n^m\right)'(x)\right)^2+2mxP_n^m(x)\left(P_n^m\right)'(x)+\frac{m^2x^2}{1-x^2}\left(P_n^m(x)\right)^2\right)dx$$

$$=-\int_{x=-1}^{1}P_n^m(x)\left(\left(1-x^2\right)\left(P_n^m\right)'(x)\right)'\,dx$$

$$-m\int_{x=-1}^{1}\left(P_n^m(x)\right)^2\,dx+\int_{x=-1}^{1}\frac{m^2x^2}{1-x^2}\left(P_n^m(x)\right)^2\,dx$$

where integration by parts has been applied to the first two terms on the right-hand side. We now use

$$\left(\left(1-x^2\right)\left(P_n^m\right)'\right)'=-\left(n(n+1)-\frac{m^2}{1-x^2}\right)P_n^m$$

from $(\natural)_n$ to get

$$\int_{x=-1}^{1}\left(P_n^{m+1}(x)\right)^2\,dx$$

$$= \int_{x=-1}^{1} \left(n(n+1) - \frac{m^2}{1-x^2} \right) \left(P_n^m(x) \right)^2 \, dx$$

$$-m \int_{x=-1}^{1} \left(P_n^m(x) \right)^2 \, dx + \int_{x=-1}^{1} \frac{m^2 x^2}{1-x^2} \left(P_n^m(x) \right)^2 \, dx$$

$$= \left(n(n+1) - m^2 - m \right) \int_{x=-1}^{1} \left(P_n^m(x) \right)^2 \, dx$$

$$= (n-m)(n+m+1) \int_{x=-1}^{1} \left(P_n^m(x) \right)^2 \, dx.$$

By induction on m, we get

$$\int_{x=-1}^{1} \left(P_n^m(x) \right)^2 \, dx = \frac{(n+m)!}{(n-m)!} \int_{x=-1}^{1} \left(P_n(x) \right)^2 \, dx,$$

because

$$\frac{(n+m+1)!}{(n-m-1)!} = (n-m)(n+m+1) \frac{(n+m)!}{(n-m)!}.$$

Finally the L^2 norm of P_n^m over $[-1,1]$ is given by

$$\left(\int_{x=-1}^{1} \left(P_n^m(x) \right)^2 \, dx \right)^{\frac{1}{2}} = \sqrt{\frac{(n+m)!}{(n-m)!} \frac{2}{2n+1}}.$$

Final Solution of the Dirichlet Problem. Now for $\lambda = n(n+1)$ and $\mu = -m^2$, we have the solution $R(r) = r^n$ for ($\sharp$) with $\lambda = n(n+1)$ and the solutions $\sin m\theta$ and $\cos m\theta$ for (&) when $\mu = -m^2$. The final step in the solution of our Dirichlet problem is to take an $\mathbb{R}$-linear combination of the special product solutions

$$R(r)\Theta(\theta)\Phi(\varphi) = r^n \, P_n^m (\cos\varphi) \begin{cases} 1 \text{ if } m = 0 \\ \cos m\theta \text{ if } m \in \mathbb{N} \\ \sin m\theta \text{ if } m \in \mathbb{N} \end{cases}$$

to form

$$u(r,\theta,\varphi) = \sum_{n=1}^{\infty} \frac{A_{n,0}}{2} r^n \, P_n (\cos\varphi) + \sum_{n=1}^{\infty} r^n \, P_n^m (\cos\varphi) \left(A_{n,m} \cos m\theta + B_{n,m} \sin m\theta \right),$$

where the constants $A_{n,m}$, $B_{m,n}$ are obtained as follows.

$$g_m(\varphi) = \frac{1}{\pi} \int_{\theta=0}^{2\pi} u_1(\theta,\varphi) \cos m\theta \, d\theta,$$

$$
\begin{aligned}
h_m(\varphi) &= \frac{1}{\pi}\int_{\theta=0}^{2\pi} u_1(\theta,\varphi)\sin m\theta\, d\theta, \\
A_{m,n} &= \frac{(n-m)!}{(n+m)!}\int_{\varphi=0}^{\pi} g_m(\varphi)\, P_n^m(\cos\varphi)\sin\varphi\, d\varphi, \\
B_{m,n} &= \frac{(n-m)!}{(n+m)!}\int_{\varphi=0}^{\pi} h_m(\varphi)\, P_n^m(\cos\varphi)\sin\varphi\, d\varphi.
\end{aligned}
$$

For the last two equations we have used the transformation $x = \cos\varphi$ with $dx = -\sin\varphi\, d\varphi$ and the interval $[-1,1]$ of the integration for x corresponding to the negative of the interval $[0,\pi]$ of the integration for φ.