Legendre Transformation

Two Ways to Define a Plane Curve. Usually a plane curve in the xy-plane is defined by $y = f(x)$. This way is to define it as the locus of a point (x, y) subject to the constraint $y = f(x)$. There is another way of defining the curve. It is to define it as the envelope of a family of straight lines $y = px - g(p)$ given in "slope-intercept" form parametrized by p. If one wants to use a description to unify these two ways of defining a curve, one can say that both ways seek a curve with a higher-order of contact (or incidence) with a family of geometric objects. For the first method, the family of geometric objects is a family of points and higher-order contact means that the point is on the curve. For the second method, the family of geometric objects is a family of straight lines and higher-order contact means that a line in the family is tangential to the curve.

Legendre Transformation to Relate the Two Ways to Define a Plane Curve. The Legendre transformation seeks to transform one way of definition of a plane to the other. Let us start out with a family of straight lines $y = px - g(p)$ given in "slope-intercept" form. Let $(x(p), y(p))$ be the point on the curve where the curve $y = f(x)$ is tangential to the straight line $y = px - g(y)$. Then

$$(*) \qquad f'(x(p)) = p \quad \text{and} \quad f(x(p)) = px(p) - g(p).$$

The first equation of $(*)$ is the "tangency" equation (which states the straight line with parameter p is tangential to the curve $y = f(x)$ at the point $x(p)$) and the second equation of $(*)$ is the "incidence" equation (which states that the point $(x(p), px(p) - g(p))$ is on the curve $y = f(x)$). Another way to express this condition is to eliminate f' from both equations by differentiating the second one. That is, we differentiate the incidence equation to eliminate f' from the tangency equation.

By regarding x as a function of p and differentiating the second equation with respect to p, we get by the chain rule

$$f'(x(p))\, x'(p) = x(p) + px'(p) - g'(p).$$

By using the first equation of $(*)$ to eliminate $f'(x(p))$, we get

$$px'(p) = x(p) + px'(p) - g'(p),$$

which means

$$x(p) - g'(p) = 0.$$

Another way to express this condition is that

$$\frac{\partial}{\partial p} G(x,p) = 0,$$

where $G(x,p) = xp - g(p)$. Thus to get $f(x)$ from $g(p)$, we can use the following recipe.

(i) Form $G(x,p) = xp - g(p)$ from the "negative-intercept" $g(p)$. Note that this function is actually the ordinate y of the point with abscissa x of the straight line in "slope-intercept" form for parameter p.

(ii) Write down the condition $\frac{\partial}{\partial p} G(x,p) = 0$ of the vanishing of the partial derivative of $G(x,p)$ with respect to the parameter p. This gives us a relation $p = p(x)$ after we solve for p in terms of x.

(iii) Finally the function $f(x)$ (for the curve $y = f(x)$) is given by $G(x, p(x))$.

Involutive Property of Legendre Transformation. From the last step of the recipe we see that the auxiliary function $G(x,p)$ is actually $f(x)$. So in the first step of the recipe, we can write $xp = f(x) + g(p)$. This makes the rôles of (x, f) and (p, g) symmetric. When we start out with $g(p)$ and we perform a Legendre transform according to the above three-step recipe, we get $f(x)$. Now we can start out with $f(x)$ (*i.e.,* g being replaced by f and p being replaced by x) and we we perform a Legendre transform according to the above three-step recipe, we are actually getting back $g(p)$ for the following reason. Since we start with (f, x) instead of (g, p), instead of $G(x,p) = xp - g(p)$ we should now form $F(p,x) = px - f(x)$ as the first step in our three-step recipe. For the second step in the three-step recipe, we write down $\frac{\partial}{\partial x} F(p,x) = 0$ (which means $p = f'(x)$) and solve for x in terms of p to get $x = x(p)$. Finally we form $px(p) - f(x(p))$ and we would like to check that indeed this result $px(p) - f(x(p))$ is equal to $g(p)$. This is equivalent to the following. When we have the relation

$$xp = f(x) + g(p) \tag{‡}$$

between x and p, the condition

$$\frac{\partial}{\partial p} G(x,p) = x - g'(p) = 0 \tag{†}$$

is equivalent to the condition

$$\frac{\partial}{\partial x}F(p,x) = p - f'(x) = 0. \qquad (\dagger\dagger)$$

This is clear when we take the differential of both sides of (‡) and get

$$x dp + p dx = f'(x) dx + g'(p) dp$$

which can be rewritten as

$$\left(x - g'(p)\right) dp + \left(p - f'(x)\right) dx = 0$$

so that $x - g'(p) = 0$ if and only if $p - f'(x) = 0$. (The reason for taking the total differentiation is to put in symmetric form the two procedures of differentiating with respect to each of the two variables.) When a transformation is applied twice to give back the identity transformation, the transformation is called *involutive*. In this sense the Legendre transformation is involutive.

Remark on the Construction of the Envelope Curve for a Given System of Curves. Suppose we have a system of curves $f(x,y,c) = 0$ with parameter c and we are interested in finding the envelope for the system. Let $x = x(t)$, $y = y(t)$ be the envelope curve in parametric form. Then at each point $(x(t),\, y(t))$ on the envelope curve there is some $c(t)$ so that the curve $f(x,y,c) = 0$ with $c = c(t)$ is tangential to the envelope curve $x = x(t)$, $y = y(t)$. This statement can be expressed in terms of two equations. One is the incidence equation $f\left(x(t), y(t), c(t)\right) \equiv 0$ in t which states that the point $(x(t),\, y(t))$ is on the curve $f(x,y,c) = 0$ with $c = c(t)$. The other is the tangency equation

$$(\dot{x}(t), \dot{y}(t)) \cdot \operatorname{grad}_{x,y} f\left(x(t), y(t), c(t)\right) = 0, \qquad (\$)$$

which states that the tangent direction $(\dot{x}(t), \dot{x}(t))$ of the envelop curve at the point $(x(t),\, y(t))$ is tangential to the curve $f(x,y,c) = 0$ with $c = c(t)$ at the point $(x(t),\, y(t))$. Here the dot above x and y means differentiation with respect to t. We now differentiate the incidence equation with respect to t to eliminate the gradient of f with respect to x, y. The differentiation of the incidence equation with respect to t yields

$$(\dot{x}(t), \dot{y}(t)) \cdot \operatorname{grad}_{x,y} f\left(x(t), y(t), c(t)\right) + f_c\left(x(t), y(t), c(t)\right) c'(t) = 0.$$

This together with ($) gives $f_c\left(x(t), y(t), c(t)\right) c'(t) = 0$. When we assume that the parameter c is chosen with $c'(t) \neq 0$, we get $f_c\left(x(t), y(t), c(t)\right) = 0$. This together with the incidence equation yields the following two equations

$$f(x, y, c) = 0, \quad f_c(x, y, c) = 0$$

which gives us an equation for the envelope curve after we eliminate c from them. This general procedure of differentiating the incidence equation and using elimination of the derivative from the tangency equation tells us that the seemingly *ad hoc* method which we used above to introduce the Legendre transformation actually simply follows the general construction of the envelope curve for a given 1-parameter family of curves.

Lagrangian and Hamiltonian Related by Legendre Transformation. Instead of using $F\left(x, y, y'\right)$, we now use the notation $L\left(t, q, \dot{q}\right)$, where L stands for the Lagrangian (to be explained later as the kinetic energy minus the potential energy in the context of a system of N particles in $\mathbb{R}^3$) and t stands for time and q for a generalized coordinate (or an n-tuple of generalized coordinates in the vector notation) and $\dot{q}$ for the derivative of q with respect to t.

We are going to apply the Legendre transformation to the Lagrangian $L\left(q, \dot{q}\right)$ which is regarded as a function of $\dot{q}$ (with q being regarded as a constant so far as the Legendre transformation is concerned). In other words, we consider the case with the Euclidean coordinate as a variable replaced by the velocity $\dot{q}$ of the generalized coordinate q. We apply the Legendre transformation to L and get g which we denote by $H = H\left(q, p\right)$, with p as the new variable replacing $\dot{q}$ (again with q being regarded as a constant so far as the Legendre transformation is concerned). Then we have

$$(\flat) \qquad H = \dot{q}p - L.$$

The variables t, q, p, H are called *canonical variables.*

The condition which governs the Legendre transformation is one of the two equivalent equations (which one to use depending on which direction we apply the involutive Legendre transformation).

$$\dot{q} = \frac{\partial}{\partial p} H\left(q, p\right),$$
$$p = \frac{\partial}{\partial \dot{q}} L\left(q, \dot{q}\right).$$

For the case at hand, we start out with L to get H. So the second equation

$$(\natural) \qquad p = \frac{\partial}{\partial \dot{q}} L(q, \dot{q})$$

is to be used. The Euler-Lagrange equation for the Lagrangian is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

By differentiating ($\flat$) with respect to q to get

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$$

and using ($\natural$) to replace

$$\frac{\partial}{\partial \dot{q}} L(q, \dot{q})$$

by p, we can rewrite the Euler-Lagrange equation as

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

This together with the first equation

$$\dot{q} = \frac{\partial}{\partial p} H(q, p)$$

from the system of two equivalent equations forming the condition governing the Legendre equation, we obtain the following Hamiltonian equations

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \end{aligned}$$

as an alternative formulation of the Euler-Lagrange equation. The Hamiltonian equations are also known as the canonical equations.

Motivation for Legendre Transformation and Hamiltonian Equations. The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

came from the Newtonian equations of motion and is a second-order differential equation. One simple way of reducing a second-order differential equation to a system of first-order differential equations is to regards the first derivative of the dependent variable as a new independent variable. For the case of the Euler-Lagrange equation it would be easier to write down the first-order differential equations by introducing

$$p = \frac{\partial L}{\partial \dot{q}}$$

as the new independent variable so that the Euler-Lagrange equation becomes

$$(\%) \qquad \frac{dp}{dt} = \frac{\partial L}{\partial q}.$$

The new problem is that we have to relate the new variable p to $\dot{q}$ in order to get rid of the second-order differentiation. That is why the Legendre transformation comes into the picture which relates p to $\dot{q}$ precisely through the equation (%). We solve for $\dot{q}$ in (%) in terms of p to get the other first-order differential equation. This is precisely the steps in the recipe for the Legendre transformation. The Hamiltonian H comes just from expressing L in terms of the new variables (q, p) instead of $(q, \dot{q})$. However, it turns out that $H = \dot{q}p - L$ is a more natural form than L in the sense that the pair (p, H) is dual to the pair $(\dot{q}, L)$.

Liouville's Theorem on Conversation of Volume in the Phase Space for Motion with Time-Independent Hamiltonian. Consider the case when the Hamiltonian $H(q, p)$ is independent of t. Let $(x_1, x_2) = (q, p)$ and

$$(\sharp) \qquad \xi_1 = \frac{\partial H}{\partial p} \quad \text{and} \quad \xi_2 = -\frac{\partial H}{\partial q}.$$

The space of the variables (q, p) is called the *phase space*. The trajectory of motion in the phase space is given by the Hamiltonian equations, which in our new notations is

$$\frac{dx_j}{dt} = \xi_j\left(x_1, x_2\right) \quad \text{for} \quad j = 1, 2.$$

Let $a_j = x_j(0)$ be the initial values of x_j at time $t = 0$ for $j = 1, 2$. By calculating the higher-order derivatives with respect to t from the differential equations and write down the Taylor expansions of $x_j(t)$ at the point $t = 0$, we get

$$x_j(t) = a_j + \xi_j(a_1, a_2)\, t + O\left(t^2\right) \quad \text{for} \quad j = 1, 2.$$

For fixed t we form the Jacobian determinant $J(a_1, a_2, t)$ of the coordinate transformation

$$(a_1,\, a_2) \mapsto (x_1(t),\, x_2(t))$$

and get

$$J(a_1, a_2, t) = \begin{vmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} \\ \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \frac{\partial \xi_1}{\partial a_1} t + O(t^2) & \frac{\partial \xi_1}{\partial a_2} t + O(t^2) \\ \\ \frac{\partial \xi_2}{\partial a_1} t + O(t^2) & 1 + \frac{\partial \xi_2}{\partial a_2} t + O(t^2) \end{vmatrix}$$

$$= 1 + \left(\frac{\partial \xi_1}{\partial a_1} + \frac{\partial \xi_2}{\partial a_2}\right) t + O\left(t^2\right).$$

Differentiating with respect to t, we get

$$\left.\frac{d}{dt} J(a_1, a_2, t)\right|_{t=0} = \frac{\partial \xi_1}{\partial a_1} + \frac{\partial \xi_2}{\partial a_2} = \operatorname{div}(\xi_1, \xi_2).$$

Suppose a domain D_0 in the phase space moves to the domain D_t after a lapse of time t according the motion given by the Hamiltonian equations. Then the volume of D_0 is given by

$$\int_{(a_1, a_2) \in D_0} da_1 \wedge da_2$$

whereas the volume of D_t is given by

$$\int_{(a_1, a_2) \in D_t} da_1 \wedge da_2 = \int_{(a_1, a_2) \in D_0} J(a_1, a_2, t)\, da_1 \wedge da_2.$$

The rate of change of the volume at time $t = 0$ is given by

$$\lim_{t \to 0} \frac{1}{t} \left(\int_{(a_1, a_2) \in D_t} da_1 \wedge da_2 - \int_{(a_1, a_2) \in D_0} da_1 \wedge da_2 \right)$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{1}{t} \left(\int_{(a_1,a_2)\in D_0} J(a_1,a_2,t)\, da_1 \wedge da_2 - \int_{(a_1,a_2)\in D_0} da_1 \wedge da_2 \right) \\
&= \int_{(a_1,a_2)\in D_0} \left(\frac{d}{dt} J(a_1,a_2,t) \right)_{t=0} da_1 \wedge da_2 \\
&= \int_{(a_1,a_2)\in D_0} \left(\frac{d}{dt} \operatorname{div}(\xi_1,\xi_2) \right)_{t=0} da_1 \wedge da_2
\end{aligned}
$$

For our vector field at hand defined by ($\sharp$), its divergence is computed by

$$
\begin{aligned}
\operatorname{div}(\xi_1,\xi_2) &= \operatorname{div}\left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q} \right) \\
&= \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p} \right) + \frac{\partial}{\partial q}\left(-\frac{\partial H}{\partial q} \right) = 0.
\end{aligned}
$$

Thus we have proved the theorem of Liouville that when the Hamiltonian is independent of time t, the volume is conserved in the phase space under the motion given by the Hamiltonian equations.

Another way to say it is that the volume form $dq \wedge dp$ is conserved under the motion given by the Hamiltonian equations. The form $dq \wedge dp$ which in the case of many variables is

$$dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n$$

is known as a symplectic form, characterized by its being a differential 2-form whose determinant is nowhere zero. Any differential 2-form whose determinant is nowhere zero can be put into the form

$$dq_1 \wedge dp_1 + \cdots + dq_n \wedge dp_n$$

at any prescribed point for some local coordinates $(q_1, p_1, \cdots, q_n, p_n)$ centered at that point.

Interpretation of Hamiltonian as Energy. Consider the case of a number of N particles of mass m_j at position $\mathbf{x}_j$ in $\mathbb{R}^3$ for $1 \leq j \leq N$. Suppose we have a number of constraints on the system of N particles so that we end up with n degrees of freedom with n parameters $q_1, \cdots, q_n$ which are the generalized coordinates. The motion of the j-th particle is given by

$$\mathbf{x}_j(t) = \mathbf{x}_j\left(q_1(t), \cdots, q_n(t)\right)$$

with its dependence on time t only through the generalized coordinates $q_1(t), \cdots, q_n(t)$. We use a dot on top of a symbol to denote its derivative with respect to the time variable t. Then by the chain rule

$$(\diamondsuit) \qquad \dot{\mathbf{x}}_j = \sum_{\nu=1}^{n} \frac{\partial \mathbf{x}_j}{\partial q_\nu}\, \dot{q}_\nu$$

and

$$(\heartsuit) \qquad \frac{d}{dt}\frac{\partial \mathbf{x}_j}{\partial q_\nu} = \sum_{\mu=1}^{n} \frac{\partial^2 \mathbf{x}_j}{\partial q_\mu \partial q_\nu}\, \dot{q}_\mu$$

For the Euler-Lagrange equation, in calculation of the derivative of the integrand in the integral functional with respect to y and y', the two quantities y and y' are treated as independent variables. They now correspond to q_ν and $\dot{q}_\nu$. From $(\diamondsuit)$ we get

$$(\diamondsuit)' \qquad \frac{\partial \dot{\mathbf{x}}_j}{\partial \dot{q}_\nu} = \frac{\partial \mathbf{x}_j}{\partial q_\nu}.$$

We also get from $(\diamondsuit)$

$$(\diamondsuit)'' \qquad \frac{\partial \dot{\mathbf{x}}_j}{\partial q_\nu} = \sum_{\mu=1}^{n} \frac{\partial^2 \mathbf{x}_j}{\partial q_\mu \partial q_\nu}\, \dot{q}_\mu,$$

which on account of $(\heartsuit)$ yields

$$(\heartsuit)' \qquad \frac{\partial \dot{\mathbf{x}}_j}{\partial q_\nu} = \frac{d}{dt}\frac{\partial \mathbf{x}_j}{\partial q_\nu}.$$

The kinetic energy T of the system of N particles is given by

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j\, \dot{\mathbf{x}}_j \cdot \dot{\mathbf{x}}_j,$$

from which we get

$$\frac{\partial T}{\partial \dot{q}_\nu} = \sum_{j=1}^{N} m_j\, \dot{\mathbf{x}}_j \cdot \frac{\partial \dot{\mathbf{x}}_j}{\partial \dot{q}_\nu} = \sum_{j=1}^{N} m_j\, \dot{\mathbf{x}}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_\nu},$$

because of $(\diamondsuit)'$, and we get

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu} = \sum_{j=1}^N m_j\, \ddot{\mathbf{x}}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_\nu} + \sum_{j=1}^N m_j\, \dot{\mathbf{x}}_j \cdot \frac{d}{dt}\frac{\partial \mathbf{x}_j}{\partial q_\nu}.$$

which on account of $(\heartsuit)'$ yields

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu} = \sum_{j=1}^N m_j\, \ddot{\mathbf{x}}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_\nu} + \sum_{j=1}^N m_j\, \dot{\mathbf{x}}_j \cdot \frac{d}{dt}\frac{\partial \mathbf{x}_j}{\partial q_\nu},$$

Also we have

$$\frac{\partial T}{\partial q_\nu} = \sum_{j=1}^N m_j\, \dot{\mathbf{x}}_j \cdot \frac{\partial \dot{\mathbf{x}}_j}{\partial q_\nu},$$

which on account of $(\heartsuit)'$ yields

$$\frac{\partial T}{\partial q_\nu} = \sum_{j=1}^N m_j\, \dot{\mathbf{x}}_j \cdot \frac{d}{dt}\frac{\partial \mathbf{x}_j}{\partial q_\nu}$$

and

$$(\clubsuit) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu} - \frac{\partial T}{\partial q_\nu} = \sum_{j=1}^N m_j\, \ddot{\mathbf{x}}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_\nu}.$$

Let $V = V(\mathbf{x}_1, \cdots, \mathbf{x}_N)$ be the potential function so that by the second law of Newtonian mechanics

$$m_j\, \ddot{\mathbf{x}}_j = -\mathrm{grad}_{\mathbf{x}_j} V$$

and

$$\sum_{j=1}^N m_j\, \ddot{\mathbf{x}}_j \cdot \frac{\partial \mathbf{x}_j}{\partial q_\nu} = -\sum_{j=1}^N \mathrm{grad}_{\mathbf{x}_j} V \cdot \frac{\partial \mathbf{x}_j}{\partial q_\nu} = -\frac{\partial V}{\partial q_\nu},$$

which together with $(\clubsuit)$ gives

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\nu} - \frac{\partial T}{\partial q_\nu} = -\frac{\partial V}{\partial q_\nu}$$

or equivalently, due to the independence of V on $\dot q_1, \cdots, \dot q_n$, the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_\nu} - \frac{\partial L}{\partial q_\nu} = 0,$$

of finding the extremal for the functional

$$\int_{t=t_1}^{t_2} L\left(t, q_1, \cdots, q_n, \dot q_1, \cdots, \dot q_n\right)\, dt,$$

where $L = T - V$. Now let us write down the Hamiltonian

$$H = \sum_{\nu=1}^{n} \dot q_\nu L_{\dot q_\nu} - L.$$

Lrom the definition of $L = T - V$ we have

$$(\spadesuit) \qquad H = \sum_{\nu=1}^{n} \dot q_\nu \left(T - V\right)_{\dot q_\nu} - \left(T - V\right) = \sum_{\nu=1}^{n} \dot q_\nu T_{\dot q_\nu} - T + V.$$

Lrom the definition of

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j\, \dot{\mathbf{x}}_j \cdot \dot{\mathbf{x}}_j$$

we get

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j \left(\sum_{\mu=1}^{n} \frac{\partial \mathbf{x}_j}{\partial q_\mu}\, \dot q_\mu\right) \cdot \left(\sum_{\nu=1}^{n} \frac{\partial \mathbf{x}_j}{\partial q_\nu}\, \dot q_\nu\right)$$

and conclude that T is a homogeneous polynomial of degree two in the variables $\dot q_1, \cdots, \dot q_n$. Thus by Euler's formula for derivatives of homogeneous polynomials we get

$$\sum_{\nu=1}^{n} \dot q_\nu T_{\dot q_\nu} = 2T$$

and $(\spadesuit)$ yields

$$H = 2T - T + V = T + V$$

which is the sum of the kinetic energy and potential energy for the system of N particles under $N - n$ constraints. This gives us the interpretation of the Hamiltonian H as energy.

Conservation of Energy in Steady System. When $L(t,q,q')$ is independent of t, the Hamiltonian $H(t,q,p) = q'p - L(t,q,q')$ with $p = L_{q'}$ is also independent of x. Then along an extremal which satisfies the canonical equation

$$\frac{dq}{dt} = \frac{\partial H}{\partial p},$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

we have (on account of the vanishing of the partial derivative of H with respect to t)

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{dq}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0,$$

giving us the conservation of the energy H along an extremal. This is the motivation for getting the first integral $F - y'F_{y'} =$ constant in the case when the integrand $F(x,y,y')$ in the integral functional

$$J = \int_{x=x_1}^{x_2} F(x,y,y')$$

is independent of x.