Maximum Principle

Let u be a real-valued harmonic function on a domain Ω in $\mathbb{C}$. One property which a harmonic function enjoys is the *mean-value property* which states that

$$(*) \qquad u(a) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u\left(a + re^{i\theta}\right) d\theta$$

whenever $\{ |z-a| \leq r \}$ is inside Ω. The reason is as follows. Take any $\tilde{r} > r$ such that $\{ |z-a| \leq \tilde{r} \}$ is inside Ω. Then u is the real part of some holomorphic function f on $\{ |z-a| \leq \tilde{r} \}$. By the Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)\, dz}{z-a}$$

which, with the parametrization $z = a + re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, yields

$$f(a) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f\left(a + re^{i\theta}\right) d\theta.$$

Now the mean-value property equation $(*)$ comes from taking the real part of both sides of this equation.

Remark. A real-valued function which satisfies the mean-valued property must be harmonic, but we will not discuss this statement here.

For any real-valued continuous function which enjoys the mean-value property, the following *maximum principle* holds.

Maximum Principle. Suppose D is a bounded domain in $\mathbb{C}$ and u is a real-valued continuous function on D which enjoys the mean-value property. Let $M = \sup_{z \in D} u(z)$. If there exists some $a \in D$ such that $u(a) = M$, then u must be identically constant on D.

Proof. Take any $r > 0$ such that $\{ |z-a| \leq r \}$ is inside D. From the mean-value property equation $(*)$ it follows that

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left(M - u\left(a + re^{i\theta}\right)\right) d\theta,$$

which implies that $u\left(a + re^{i\theta}\right) = M$ for all $0 \leq \theta \leq 2\pi$. Thus we conclude that if u assumes M at a point of D, it is identically constant on any disk in D centered at that point. We can now join any point z in D to a by a curve γ in D and cover γ by a finite number of open disks $\Delta_1, \cdots, \Delta_k$ such that a is the center of Δ_1 and the center of Δ_j is inside D_{j-1} for $2 \leq j \leq k$ and the point z is inside D_k. Then we can conclude that u is identically equal to M on each D_j for $1 \leq j \leq k$ and, in particular, $u(z) = M$. Q.E.D.

Harmonic Function Uniquely Determined by Boundary Value. When we solve the Dirichlet problem of finding a harmonic function in the interior with prescribed boundary values, we did not discuss the question of uniqueness. For a bounded domain, when the boundary value of a harmonic function is zero, the harmonic function must be zero, because we can apply the maximum principle both to the harmonic function and its negative. Thus, if there are two solutions of the Dirichlet problem, we can take their difference and conclude that the vanishing of the boundary value of the difference implies that the vanishing of the difference in the interior and the two solutions must be the same.

Uniqueness for Neumann Boundary Value Problem from the Divergence Theorem. If a solution of the Laplace equation is to satisfy partially a boundary value condition and partially a Neumann condition of the vanishing of the normal derivative at the boundary, we can get uniqueness from the divergence theorem. Suppose u is harmonic on a bounded domain D so that at any point of the boundary of D either the boundary value of u is zero or the normal derivative of u is zero. From

$$\operatorname{div}\left(u \operatorname{grad} u\right) = \left|\operatorname{grad} u\right|^2 + u\,\Delta u$$

and the application of the divergence theorem to the vector field $u \operatorname{grad} u$ on D, we conclude that

$$\int_{\partial D} u \operatorname{grad} u = \int_D \operatorname{div}\left(u \operatorname{grad} u\right) = \int_D \left(\left|\operatorname{grad} u\right|^2 + u\,\Delta u\right)$$

and

$$\int_D \left|\operatorname{grad} u\right|^2 = 0.$$

Thus the gradient of u is identically zero on D and u is constant. If the boundary value of u is zero somewhere, then u must be identically zero.

Remark on Unbounded Domain. For the uniqueness question for a solution of the Laplace equation with prescribed Dirichlet or Neumann boundary values on an unbounded simply connected domain D, we apply the Riemann mapping theorem (which was mentioned in this course but not proved) to conformally map the domain to the open unit disk and apply the above uniqueness to the open unit disk. For the case of fluid flow, when the total flux is finite across infinite openings, the jump in the values of streamline function across the infinite openings are finite. When we apply the uniqueness arguments, we have to use a dent around the point on the unit circle which corresponds to the infinity point.