Noether's Theorem on First Integrals From Transformations Leaving the Functional Invariant

Consider the variation problem of the functional

$$J = \int_{x_1}^{x_2} F\left(x, y, y'\right) dx.$$

Assume that we have a 1-parameter family of transformations

$$(\dagger) \qquad x^* = \Phi\left(x, y, y', \varepsilon\right), \quad y^* = \Psi\left(x, y, y', \varepsilon\right)$$

(parametrized by ε) which leave the functional J invariant. We assume that when ε is 0 the transformation (†) is just the identity transformation. We are going to introduce Noether's theorem which will yield a first integral of the extremal from the 1-parameter family of transformations which leave J invariant.

Note that for any fixed ε the transformation (†) is not a transformation of points. When we have a curve C given by $y = y(x)$, the transformation (†) for any fixed ε yields a curve C_ε by representing x^* as a function $\Phi\left(x, y(x), y'(x), \varepsilon\right)$ of x and representing y^* as a function $\Psi\left(x, y(x), y'(x), \varepsilon\right)$ of x so that C_ε is given in the parametric form $x^* = x^*(x)$ and $y^* = y^*(x)$ with x as the parameter. To get the point (x^*, y^*) we need to know not just the point (x, y) but also the derivative y'. The motivation for such a formulation is that it is more natural to consider the triple (x, y, p) than the pair (x, y) because of the canonical differential equations, but we cannot just introduce a transformation which sends (x, y, p) to (x^*, y^*, p^*), because p^* is already determined when y^* is given as a function of x^*. This consideration makes it necessary to consider a transformation of the form (†) which sends a curve C to a curve C_ε, but not a point (x, y) to a point (x^*, y^*).

The condition that J is invariant under (†) for every fixed ε means that J evaluated at C is equal to J evaluated at C_ε.

Assume that C given by $y = y(x)$ is an extremal for the variational problem for the functional J. We now consider the family of curves C_ε parametrized by ε. Note that since C_ε is given in parametrized form with x as the parameter, in general the two end-points $\left(x^*\left(x_1\right), y^*\left(x_1\right)\right)$ and $\left(x^*\left(x_2\right), y^*\left(x_2\right)\right)$ of C_ε vary when ε varies. We now apply the general variation formula to this family of curves C_ε. Since J is invariant under the transformation (†), it follows from C being an extremal for J that each C_ε is also an extremal for J and satisfies the Euler-Lagrange equation. Thus the general variation formula (evaluated at $\varepsilon = 0$) for this family of curves C_ε is given by

$$0 = \delta J = F_{y'}\delta y + \left(F - y'F_{y'}\right)\delta x\Big|_{x=x_1}^{x_2},$$

where (x_1, y_1) and (x_2, y_2) (with $y_1 = y(x_1)$ and $y_2 = y(x_2)$) are the two end-points of $C = C_0$ and $\delta x = \frac{\partial x^*}{\partial \varepsilon}\Big|_{\varepsilon=0}$ and $\delta y = \frac{\partial y^*}{\partial \varepsilon}\Big|_{\varepsilon=0}$. We rewrite it as

$$F_{y'}\delta y + \left(F - y'F_{y'}\right)\delta x\Big|_{x=x_1} = F_{y'}\delta y + \left(F - y'F_{y'}\right)\delta x\Big|_{x=x_2}$$

for any choices of x_1 and x_2. Now we fix x_1 and let $x_2 = x$ vary. Then the value of

$$F_{y'}\delta y + \left(F - y'F_{y'}\right)\delta x$$

at x is equal to its value at x_1 which is a constant independent of x. In other words, the expression

$$F_{y'}\delta y + \left(F - y'F_{y'}\right)\delta x$$

is equal to a constant along the extremal. This is Noether's theorem. Its precise formulation is that

$$F_{y'}\psi + \left(F - y'F_{y'}\right)\varphi = \text{constant}$$

along the extremal, where

$$\varphi = \frac{\partial}{\partial \varepsilon}\Phi\left(x, y(x), y'(x), \varepsilon\right)\Big|_{\varepsilon=0},$$
$$\psi = \frac{\partial}{\partial \varepsilon}\Psi\left(x, y(x), y'(x), \varepsilon\right)\Big|_{\varepsilon=0}.$$

The first integral

$$F - y'F_{y'} = \text{constant}$$

for the case where $F(x, y, y')$ is independent of x is the special case of Noether's theorem where

$$\begin{cases} \Phi\left(x, y(x), y'(x), \varepsilon\right) = x + \varepsilon, \\ \\ \Psi\left(x, y(x), y'(x), \varepsilon\right) = y \end{cases}$$

so that $\varphi = 1$ and $\psi = 0$.