Derivation of the Poisson Kernel
From the Cauchy Formula

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The Cauchy's integral formula states that

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta) d\zeta}{\zeta - z}$$

for $|z| < 1$ if f is holomorphic on the open unit 1-disk and continuous up to the boundary. We want to make the kernel real so that we can get a kernel for harmonic functions by taking the real part of the formula. Let $\zeta = e^{i\theta}$. Then

$$d\zeta = i\, e^{i\theta} d\theta = i\, \zeta\, d\theta$$

and

$$(\sharp) \qquad f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \frac{\zeta}{\zeta - z} d\theta.$$

So we try to make $\frac{\zeta}{\zeta - z}$ real. The easiest way is to take the complex conjugate

$$\frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} = \frac{\zeta \bar{\zeta}}{\zeta \bar{\zeta} - \zeta \bar{z}} = \frac{1}{1 - \zeta \bar{z}} = \frac{\zeta}{\zeta (1 - \zeta \bar{z})}$$

and

$$\frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} d\theta = \frac{1}{i} \frac{i\zeta}{\zeta(1 - \zeta \bar{z})} d\theta = \frac{1}{i} \frac{d\zeta}{\zeta(1 - \zeta \bar{z})}.$$

Hence

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} d\theta = \frac{1}{2\pi i} \int_{\theta=0}^{2\pi} \frac{f(\zeta) d\zeta}{\zeta(1 - \zeta \bar{z})} = f(0).$$

Since

$$f(0) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) d\theta,$$

it follows that

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left(\frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} - 1 \right) f(\zeta) = 0.$$

We add this to $(\sharp)$ and obtain

$$f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} - 1 \right) d\theta$$

We now rewrite the reproducing kernel as

$$\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - \bar{z}} - 1 = 2\,\mathrm{Re}\,\left(\frac{\zeta}{\zeta - z} - \frac{1}{2}\right) = 2\,\mathrm{Re}\,\frac{2\zeta - \zeta + z}{2(\zeta - z)} = \mathrm{Re}\,\frac{\zeta + z}{\zeta - z}.$$

This is the Poisson kernel. We can also write

$$\mathrm{Re}\frac{\zeta + z}{\zeta - z} = \mathrm{Re}\frac{\bar{\zeta}\zeta - \bar{z}\zeta + z\bar{\zeta} - z\bar{z}}{|\zeta - z|^2} = \frac{\zeta\bar{\zeta} - z\bar{z}}{|\zeta - z|^2}.$$

The Poisson integral formula is

$$f(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f(\zeta)\,\mathrm{Re}\,\frac{\zeta + z}{\zeta - z}\,d\theta$$

or equivalently

$$f(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f(\zeta)\,\frac{\zeta\bar{\zeta} - z\bar{z}}{|\zeta - z|^2}\,d\theta.$$