

### Derivation of the Poisson Kernel From the Method of Electrostatics

First we introduce the concept of Green's function for a bounded domain in  $\mathbf{C}$ .

*Definition.* The Green's function  $G(\zeta, x)$  for a bounded domain  $\Omega$  in  $\mathbf{C}$  with smooth boundary is a function on  $\overline{\Omega} \times \Omega$ , where  $\overline{\Omega}$  is the topological closure of  $\Omega$ , which satisfies the following two conditions.

- (i) For fixed  $z \in \Omega$  the Laplacian of  $G(\zeta, z)$  with respect to  $\zeta$  is the Dirac delta function  $\delta_z(\zeta)$  at  $z$ .
- (ii)  $G(\zeta, z)$  is identically zero for  $\zeta \in \partial\Omega$ .

In terms of electrostatics this means that  $G(\zeta, z)$  is the electrostatic potential due to a point charge at the point  $z$  which is normalized to be zero at the boundary. We can also interpret the boundary as a conductor so that the potential at every point of the boundary is a constant which we normalize to be zero. Since the context of our discussion is real dimension two, in physical reality this means that we have a wire inside, and parallel to, a conducting hollow cylinder (with base  $\Omega$ ) so that the wire has uniform charge density on it. The Green's function is simply the electrostatic potential between the wire and the conducting cylinder.

*Application of the Divergence Theorem to Obtain the Poisson Kernel from Green's Function.* Let  $f$  and  $g$  be two real-valued functions on  $\overline{\Omega}$  with order of differentiability sufficient for the discussion below. From the divergence theorem applied to

$$\operatorname{div}(f \operatorname{grad} g) = (\operatorname{grad} f) \cdot (\operatorname{grad} g) + f \Delta g$$

we get

$$\int_{\partial\Omega} f \operatorname{grad}_n g = \int_{\Omega} (\operatorname{grad} f) \cdot (\operatorname{grad} g) + \int_{\Omega} f \Delta g$$

where  $\operatorname{grad}_n g$  means the normal derivative. Reversing the roles of  $f$  and  $g$  and taking the difference of the two equations we get

$$\int_{\partial\Omega} (f \operatorname{grad}_n g - g \operatorname{grad}_n f) = \int_{\Omega} (f \Delta g - g \Delta f).$$

When  $g(\zeta) = G(\zeta, z)$ , we have

$$f(z) = \int_{\zeta \in \partial\Omega} f(\zeta) \operatorname{grad}_{\zeta, n} G(\zeta, z) + \int_{\Omega} G(\zeta, z) \Delta_{\zeta} f(\zeta).$$

For a harmonic  $f(z)$  we have the Poisson formula

$$f(z) = \int_{\zeta \in \partial\Omega} f(\zeta) \operatorname{grad}_{\zeta, n} G(\zeta, z).$$

So we know that  $\operatorname{grad}_{\zeta, n} G(\zeta, z)$  is the Poisson kernel.

*Interpretation of Normal Derivative of Green's Function in Terms of Electrostatics.* In terms of electrostatics the meaning of  $\operatorname{grad}_{\zeta, n} G(\zeta, z)$  is as follows. On the surface of the conducting cylinder there is some charge of opposite sign distributed on it because of the charge on the wire. When we apply the law of Gauss on a small rectangular cylinder centered on the conducting cylindrical surface, we conclude that  $\operatorname{grad}_{\zeta, n} G(\zeta, z)$  is simply the charge density on the cylindrical surface.

*Construction of Green's Function from Point Charges.* We now explicitly construct the Green's function. We put a point charge at  $z$  and then another point charge of opposite sign at the inversion  $\frac{1}{\bar{z}}$  of  $z$  with respect to the unit circle and we are going to find the magnitudes of the charges at the two locations so that the electrostatic potential along the unit circle is zero or constant. The potential will be of the form

$$\alpha \log |\zeta - z| - \beta \log \left| \zeta - \frac{1}{\bar{z}} \right| + \gamma$$

which is the same as

$$\alpha \log |\zeta - z| - \beta \log |\bar{z}\zeta - 1| + \beta \log |z|.$$

We have to decide what values we should choose for  $\alpha$  and  $\beta$ . Recall that  $\frac{\zeta - z}{1 - \bar{z}\zeta}$  is a fractional linear transformation mapping the unit disk to itself, because for  $|\zeta| = 1$  we have

$$\left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right| = \left| \frac{(\zeta - z)}{\bar{z}(\bar{z}\zeta - 1)} \right| = \left| \frac{\zeta - z}{\bar{z} - \bar{z}} \right| = 1.$$

Thus  $\log \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right|$  has value 0 on the unit circle. A choice which we can use is  $\alpha = \beta = -\gamma = 1$ . This means putting a unit charge at  $z$  and a unit charge of opposite sign at  $\frac{1}{\bar{z}}$  and then adjust the resulting electrostatic potential by the constant  $-1$ . So the Green's function is  $\log \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right|$ . This is to be expected, because Green's function  $G(\zeta, 0)$  is clearly  $\log |\zeta|$  and we have the invariance of Green's function under conformal mappings. We now want to compute the normal derivative of  $\log \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right|$  at the unit circle. Since at a point on the unit circle  $\frac{\partial}{\partial r} = \zeta \frac{\partial}{\partial \zeta}$  along the line through the origin, it follows that

$$\begin{aligned} \frac{\partial}{\partial r} \log \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right| &= \operatorname{Re} \frac{\partial}{\partial r} \log \left( \frac{\zeta - z}{1 - \bar{z}\zeta} \right) = \operatorname{Re} \zeta \frac{\partial}{\partial \zeta} \log \left( \frac{\zeta - z}{1 - \bar{z}\zeta} \right) \\ &= \operatorname{Re} \left( \frac{\zeta}{\zeta - z} + \frac{\zeta \bar{z}}{1 - \bar{z}\zeta} \right) = \operatorname{Re} \left( \frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}} \right) \\ &= \operatorname{Re} \left( \frac{\zeta}{\zeta - z} + \frac{z}{\zeta - z} \right) = \operatorname{Re} \frac{\zeta + z}{\zeta - z}. \end{aligned}$$

This tells us that

$$f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(\zeta) \operatorname{Re} \frac{\zeta + z}{\zeta - z} d\theta.$$