

Derivation of the Poisson Kernel by Fourier Series and Convolution

We are going to give a second derivation of the Poisson kernel by using Fourier series and convolution. We also use this derivation as an opportunity to introduce Fourier series.

Definition of Fourier Coefficients and Fourier Series. The starting point is a function $f(x)$ on \mathbb{R} with periodicity 2π (i.e., $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$) or equivalently a function $f(x)$ defined on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ or equivalently a function $f(x)$ defined on $[0, 2\pi]$ with $f(2\pi) = f(0)$. There are two ways of writing down the Fourier coefficients and Fourier series, one using complex numbers and one using only real numbers.

Use of Complex Numbers. When complex numbers are used, the Fourier coefficients $c_n = c_n(f)$ of f is defined by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \text{for } n \in \mathbb{Z}$$

and the Fourier series of f is defined as

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

First let us say something about the normalizing factor $\frac{1}{2\pi}$ used in the definition of c_n . It is the square norm of the function e^{inx} and it enters into the picture for the following reason. The functions e^{inx} for $n \in \mathbb{Z}$ are mutually orthogonal with respect to the inner product defined by the integral over $[-\pi, \pi]$ or equivalently over $[0, 2\pi]$.

$$\int_{-\pi}^{\pi} e^{imx} \overline{e^{inx}} dx = 0 \quad \text{for } m \neq n.$$

The square norm of e^{inx} is given by

$$\int_{-\pi}^{\pi} e^{inx} \overline{e^{inx}} dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

In order to make the collection $\{e^{inx}\}_{n \in \mathbb{Z}}$ into an orthonormal set, we have to form $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n \in \mathbb{Z}}$. In the space of functions on $[-\pi, \pi]$ with inner product

$$(g, h) = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx,$$

when we consider the approximation of f by a \mathbb{C} -linear combination of elements of $\left\{\frac{1}{\sqrt{2\pi}} e^{inx}\right\}_{n \in \mathbb{Z}}$, we form

$$\sum_{n=-\infty}^{\infty} \left(f, \frac{1}{\sqrt{2\pi}} e^{inx}\right) \frac{1}{\sqrt{2\pi}} e^{inx}.$$

We can put the two factors $\frac{1}{\sqrt{2\pi}}$ together to form

$$\sum_{n=-\infty}^{\infty} \left(f, \frac{1}{2\pi} e^{inx}\right) e^{inx},$$

which is the same as

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

when we define c_n by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

with the normalizing factor $\frac{1}{2\pi}$ which actually is the square norm (e^{inx}, e^{inx}) of e^{inx} .

Convolution and Coefficientwise Product. Given two functions $f(x)$ and $g(x)$ on \mathbb{R} with periodicity 2π , we define their *convolution* by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y) dy.$$

The motivation is that the convolution is the weighted average of the translates of the first function f with the second function g as the weight function. More precisely, $f(x-y)$ as a function of x is obtained (so far as the graph is concerned) by moving the function $f(x)$ to the right by a distance equal to y so that the value at $x_0 + y$ is the same as the value at x_0 . We then multiply this translate $f(x-y)$ (as a function of x) by the weight $g(y)$. The integration over $[-\pi, \pi]$ with a factor $\frac{1}{2\pi}$ is just taking the average (with weight).

With the change of the dummy variable y to $z = x - y$ the integral defining the convolution becomes

$$(f * g)(x) = -\frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(z)g(x-z) dz$$

which by the periodicity of f and g is equal to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(z)g(x-z) dz$$

which is $(g * f)(x)$. We thus conclude that the convolution operation is commutative. Note that in our definition of the convolution of f and g the factor $\frac{1}{2\pi}$ is added, which is a matter of convention and is not always used. It is used here so that the Fourier coefficient of the convolution is equal to the product of the corresponding Fourier coefficient for the two functions.

We now compute the Fourier coefficients of $f * g$ in terms of those of f and g by using Fubini's theorem for iterated integrals.

$$\begin{aligned} c_n(f * g) &= \frac{1}{2\pi} \int_{x=-\pi}^{\pi} (f * g)(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{x=-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{y=-\pi}^{\pi} f(x-y)g(y) dy \right) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{x=-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{y=-\pi}^{\pi} f(x-y)g(y) dy \right) e^{-in(x-y)} e^{-iny} dx \\ &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{x=-\pi}^{\pi} f(x-y) e^{-in(x-y)} dx \right) g(y) e^{-iny} dy \\ &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} c_n(f) g(y) e^{-iny} dy = c_n(f) c_n(g). \end{aligned}$$

It tells us that the Fourier series of $f * g$ is the coefficientwise product of the Fourier series of f and the Fourier series of g .

Partial Sum as Convolution with the Dirichlet-Dini Kernel. A most important part of the theory of Fourier series is the question of convergence, which asks in what sense and under what condition is the n -th partial sum

$$s_n(x) = \sum_{k=-n}^n c_k e^{ikx}$$

of the Fourier series of the function $f(x)$ converges to $f(x)$. Usually the question is asked in two contexts, one is about convergence in the L^2 norm and another is pointwise convergence.

Before we answer the question of convergence, first we would like to express s_n in terms of f by using the definition of the Fourier coefficients c_n .

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^n c_k e^{ikx} = \sum_{k=-n}^n \left(\frac{1}{2\pi} \int_{y=-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{ikx} \\ &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} f(y) \left(\sum_{k=-n}^n e^{in(x-y)} \right) dy = (f * D_n)(x), \end{aligned}$$

where

$$D_n(x) = \sum_{k=-n}^n e^{ikx}$$

is the Dirichlet-Dini kernel. Sometimes a factor of $\frac{1}{2\pi}$ is added to the Dirichlet-Dini kernel when the factor $\frac{1}{2\pi}$ is not used in the definition of convolution. We now use the geometric series to sum up the finite series which defined the Dirichlet-Dini kernel to get

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin \frac{x}{2}}.$$

The verification is as follows.

$$\begin{aligned} \sum_{k=-n}^n e^{ikx} &= \sum_{k=0}^n (e^{ix})^k + \sum_{k=-n}^{-1} (e^{ix})^k \\ &= \sum_{k=0}^n (e^{ix})^k + e^{-inx} \sum_{k=0}^{n-1} (e^{ix})^k \\ &= \frac{1 - (e^{ix})^{n+1}}{1 - e^{ix}} + e^{-inx} \frac{(e^{ix})^{-n} - 1}{1 - e^{ix}} \\ &= \frac{1 - (e^{ix})^{n+1}}{1 - e^{ix}} + \frac{1 - (e^{ix})^n}{1 - e^{ix}} \\ &= \frac{(e^{ix})^{-n} - (e^{ix})^{n+1}}{1 - e^{ix}} = \frac{(e^{ix})^{-n-\frac{1}{2}} - (e^{ix})^{n+\frac{1}{2}}}{(e^{ix})^{-\frac{1}{2}} - (e^{ix})^{\frac{1}{2}}} \end{aligned}$$

$$= \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin\frac{x}{2}}.$$

We note that

$$(b) \quad \frac{1}{2\pi} \int_{x=-\pi}^{\pi} D_n(x) dx = 1$$

directly from the definition of

$$D_n(x) = \sum_{k=-n}^n e^{ikx}.$$

Bessel's Inequality for Partial Sum in Terms of Orthonormal Functions. For any square-integrable function f on $[-\pi, \pi]$, we compute the square of the L^2 norm of the difference of f and its projection onto the linear subspace spanned by the orthonormal set $\left\{\frac{1}{\sqrt{2\pi}} e^{ikx}\right\}_{-n \leq k \leq n}$ and get

$$\begin{aligned} & \left\| f - \sum_{k=-n}^n \left(f, \frac{1}{\sqrt{2\pi}} e^{ikx} \right) \frac{1}{\sqrt{2\pi}} e^{ikx} \right\|^2 \\ &= \|f\|^2 - \sum_{k=-n}^n \left| \left(f, \frac{1}{\sqrt{2\pi}} e^{ikx} \right) \right|^2 \\ &= \|f\|^2 - 2\pi \sum_{k=-n}^n |c_k|^2. \end{aligned}$$

Since the left-hand side is nonnegative, by letting $n \rightarrow \infty$ we get the following Bessel's inequality

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi} \|f\|^2.$$

As a consequence,

$$\lim_{n \rightarrow \infty} c_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} c_{-n} = 0.$$

We can formulate this conclusion directly in terms of the square-integrable function f as follows.

$$\lim_{n \rightarrow \infty} \int_{x=-\pi}^{\pi} f(x) e^{-inx} dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{x=-\pi}^{\pi} f(x) e^{inx} dx = 0.$$

We can also use the sine and cosine functions instead of the exponential functions and get

$$\lim_{n \rightarrow \infty} \int_{x=-\pi}^{\pi} f(x) \cos nx \, dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{x=-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

for any square-integrable function f on $[-\pi, \pi]$. We will later use the following consequence of these two limit statements

$$(*) \quad \lim_{n \rightarrow \infty} \int_{x=-\pi}^{\pi} f(x) \sin \left(n + \frac{1}{2} \right) x \, dx = 0$$

after we use the addition formula for the sine function

$$\sin(\varphi + \psi) = \sin \varphi \cos \psi + \cos \varphi \sin \psi.$$

Note that $(*)$ is a consequence of the ever-increasing frequency of the oscillation of the sine function $\sin \left(n + \frac{1}{2} \right) x$.

Pointwise Convergence of Fourier Series. Now we look at the question of the pointwise convergence of the n -th partial sum $s_n(x)$ of the Fourier series of a function $f(x)$. We have established that

$$s_n(x) = \frac{1}{2\pi} \int_{y=-\pi}^{\pi} \frac{\sin \left(n + \frac{1}{2} \right) (x - y)}{\sin \frac{x-y}{2}} f(y) \, dy.$$

From (‡) it follows that

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} D_n(y) f(x) \, dy \\ &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} D_n(y - x) f(x) \, dy \\ &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} \frac{\sin \left(n + \frac{1}{2} \right) (y - x)}{\sin \frac{y-x}{2}} f(x) \, dy \\ &= \frac{1}{2\pi} \int_{y=-\pi}^{\pi} \frac{\sin \left(n + \frac{1}{2} \right) (x - y)}{\sin \frac{x-y}{2}} f(x) \, dy. \end{aligned}$$

As a consequence,

$$s_n(x) - f(x) = \frac{1}{2\pi} \int_{y=-\pi}^{\pi} \frac{\sin \left(n + \frac{1}{2} \right) (x - y)}{\sin \frac{x-y}{2}} (f(y) - f(x)) \, dy.$$

We rewrite it as

$$s_n(x) - f(x) = \int_{y=-\pi}^{\pi} \sin\left(\left(n + \frac{1}{2}\right)(x - y)\right) \frac{f(y) - f(x)}{2\pi \sin \frac{x-y}{2}} dy.$$

To show that, for fixed x , the difference $s_n(x) - f(x)$ converges to 0 as $n \rightarrow \infty$, it suffices to show that the function

$$\frac{f(y) - f(x)}{2\pi \sin \frac{x-y}{2}}$$

is square-integrable as a function of $x \in [-\pi, \pi]$. For example, if f is differentiable at x and f is square integrable on $[-\pi, \pi]$, then the difference quotient

$$\frac{f(y) - f(x)}{y - x}$$

is uniformly bounded on $|y - x| < \delta$ for some $\delta > 0$ as a function of y and clearly

$$\frac{f(y) - f(x)}{2\pi \sin \frac{x-y}{2}}$$

is square-integrable on $[-\pi, \pi]$ outside of $|y - x| < \delta$ as a function of y . Thus we conclude that

$$\lim_{n \rightarrow \infty} s_n(x) = f(x).$$

We would like to remark that this pointwise convergence is not from an approximate identity argument. Instead it is from the argument of ever-increasing frequency of the oscillation of the sine function $\sin\left(n + \frac{1}{2}\right)x$. This argument involves more than the continuity of the function f . Some statement about the bound or square-integrability of the difference quotient of f has to be assumed. There is another kind of convergence, called Cesàro convergence, which is a consequence an approximate identity argument. The Cesàro convergence is the convergence of the average of consecutive partial sums.

Parseval's Identity. We now would answer the question whether Bessel's inequality is indeed an identity for a square-integrable function f . Indeed it is the case and the identity is called Parseval's identity. To prove it, we simply use the 3ε -argument and the approximation of an L^2 function in the L^2 norm by a smooth function. Given any $\varepsilon > 0$, first we construct

a smooth function $g(x)$ on \mathbb{R} with periodicity 2π such that $\|f - g\| < \varepsilon$, where $\|\cdot\|$ means L^2 -norm over the interval $[-\pi, \pi]$. Since g is smooth, by the argument used to prove pointwise convergence of the Fourier series of a smooth function, we can conclude that

$$\sum_{n=-\infty}^{\infty} |c_n(g)|^2 = \frac{1}{2\pi} \|g\|^2$$

which can be rewritten as

$$\left(\sum_{n=-\infty}^{\infty} |c_n(g)|^2 \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \|g\|.$$

In particular, we can find n sufficiently large such that

$$0 \leq \frac{1}{\sqrt{2\pi}} \|g\| - \left(\sum_{k=-n}^n |c_k(g)|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

By Bessel's inequality applied to $f - g$, we obtain

$$\sum_{k=-n}^n |c_k(f - g)|^2 \leq \frac{1}{2\pi} \|f - g\|^2$$

which can be rewritten as

$$\left(\sum_{k=-n}^n |c_k(f - g)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2\pi}} \|f - g\| < \frac{\varepsilon}{\sqrt{2\pi}}.$$

By the triangle inequality,

$$\begin{aligned} \left(\sum_{k=-n}^n |c_k(f)|^2 \right)^{\frac{1}{2}} &\geq \left(\sum_{k=-n}^n |c_k(g)|^2 \right)^{\frac{1}{2}} - \left(\sum_{k=-n}^n |c_k(f - g)|^2 \right)^{\frac{1}{2}} \\ &> \left(\frac{1}{\sqrt{2\pi}} \|g\| - \varepsilon \right) - \frac{\varepsilon}{\sqrt{2\pi}} \\ &> \left(\frac{1}{\sqrt{2\pi}} (\|f\| - \varepsilon) - \varepsilon \right) - \frac{\varepsilon}{\sqrt{2\pi}}. \end{aligned}$$

Since ε is an arbitrary positive number, we conclude that

$$\left(\sum_{k=-\infty}^{\infty} |c_k(f)|^2 \right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{2\pi}} \|f\|$$

which, together with Bessel's inequality, gives the following Parseval's identity

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \frac{1}{2\pi} \|f\|^2.$$

Remark on the Fourier Series in Terms of the Sine and Cosine Functions. When we use the sine and cosine functions instead of the exponential functions to express the Fourier series of a continuous function $f(x)$ on \mathbb{R} with period 2π , the difference is with the normalizing factors. The Fourier coefficients a_k ($0 \leq k < \infty$) and b_k ($k \geq 1$) are given by

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx,$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx.$$

Note that the normalizing factors $\frac{1}{\pi}$ are used, because

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \quad \text{for } n \geq 1$$

from using the double angle formulae

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

The n -th partial sum s_n of the Fourier series of $f(x)$ is given by

$$s_n = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx).$$

There is a factor $\frac{1}{2}$ for a_0 in the expression for the n -th partial sum s_n , because the normalizing factor in the definition of the Fourier coefficient a_0 should have been $\frac{1}{2\pi}$ instead of $\frac{1}{\pi}$ from

$$\int_{-\pi}^{\pi} \cos^2(0 \cdot x) \, dx = \int_{-\pi}^{\pi} dx = 2\pi$$

but we have used only $\frac{1}{\pi}$ for unifying the formula for all a_k for $k \geq 0$ and thus we need to insert the factor $\frac{1}{2}$ for a_0 back into the definition of the n -th partial sum s_n .

Derivation of the Poisson Kernel Fourier Series and Convolution. After our discussion about Fourier series, we now come to the derivation of the Poisson kernel from Fourier series and convolution.

Let $u(z)$ be a harmonic function on an open neighborhood of the closed unit disk. We write u as $2 \operatorname{Re} f(z)$ of some holomorphic function $f(z)$. Then using the power series expansion of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$u(re^{i\theta}) = (a_0 + \bar{a}_0) + \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \bar{a}_n r^n e^{-in\theta}$$

which is equal to the Fourier series of the convolution of

$$(a_0 + \bar{a}_0) + \sum_{n=1}^{\infty} a_n e^{in\theta} + \sum_{n=1}^{\infty} \bar{a}_n e^{-in\theta}$$

and

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta} \\ &= 1 + 2 \operatorname{Re} \sum_{n=1}^{\infty} z^n = 2 \operatorname{Re} \frac{z}{1-z} + 1 \\ &= 2 \operatorname{Re} \left(\frac{z}{1-z} - \frac{1}{2} \right) = \operatorname{Re} \frac{1+z}{1-z}, \end{aligned}$$

where $z = re^{i\theta}$. Hence

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} \operatorname{Re} \frac{1 + e^{i(\theta-\varphi)}}{1 - e^{i(\theta-\varphi)}} u(e^{i\varphi}) d\varphi$$

or

$$u(z) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} \operatorname{Re} \frac{\zeta + z}{\zeta - z} u(\zeta) d\varphi$$

with $\varphi = \arg \zeta$. This is the Poisson integral formula.