Derivation of the Poisson Kernel
From the Use of the Argument Function

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We are going to derive the Poisson kernel by using the argument function and plane Euclidean geometry. We start out with the basic building block of the Poisson kernel. It is the harmonic measure. Given two points $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle, we have the harmonic measure associated to the arc which goes from $e^{i\alpha}$ to $e^{i\beta}$ in the counterclockwise sense. We denote it by $\omega_{\alpha,\beta}$. The harmonic measure $\omega_{\alpha,\beta}$ is a function on the unit disk which is characterized by the following properties.

(i) $\omega_{\alpha,\beta}$ is harmonic on the open unit disk,

(ii) $\omega_{\alpha,\beta}$ assumes the value 1 on the arc in the counterclockwise sense between $e^{i\alpha}$ and $e^{i\beta}$, and

(iii) $\omega_{\alpha,\beta}$ assumes the value 0 on the arc in the counterclockwise sense between $e^{i\beta}$ and $e^{i\alpha}$.

The Poisson kernel is actually the harmonic measure for the case where $\alpha = \theta$ and $\beta = \theta + d\theta$ or more precisely

$$\lim_{\beta\to\alpha} \frac{\omega_{\alpha,\beta}}{\beta - \alpha}\bigg|_{\alpha=\theta}.$$

One way to get the harmonic measure $\omega_{\alpha,\beta}$ is to use the argument function. The idea is the same as the one used for the construction of the integrand for the Schwarz-Christoffel transformation, namely we use the function

$$\arg\left(\frac{z - e^{i\beta}}{z - e^{i\alpha}}\right).$$

This is a branch of a holomorphic function and is defined as the difference $\arg\left(z - e^{i\beta}\right) - \arg\left(z - e^{i\alpha}\right)$ of the two branch functions $\arg\left(z - e^{i\beta}\right)$ and $\arg\left(z - e^{i\alpha}\right)$.

The branch $\arg\left(z - e^{i\beta}\right)$ is defined by removing the cut $\left\{ re^{i\beta} \,\middle|\, r \geq 1\right\}$ from $\mathbb{C}$. We have to specify the range of the numerical value of $\arg\left(z - e^{i\beta}\right)$. Let A be the point $e^{i\alpha}$ and B be the point $e^{i\beta}$ and O be the origin. We draw a tangent line to the unit circle $\{|z| = 1\}$ at the point B and put a point T_B on it so that the direction from B to T_B represents the direction of the velocity vector at B for a point moving in counterclockwise sense along the unit circle $\{|z|=1\}$. We use the interval $(0,\pi)$ as the range for $\arg\left(z-e^{i\beta}\right)-\arg\vec{BT_B}$ for a point in $\{|z|=1\}$ different from B (that is, measuring from the direction $\vec{BT_B}$). We can choose the value $\frac{\pi}{2}+\beta$ for $\arg\vec{BT_B}$, because we can choose the value β for $\arg\vec{OB}$ and the vector direction $\arg\vec{BT_B}$ is obtained by turning the vector direction $\vec{OB}$ one right angle in the counterclockwise sense. So we can use the range $\beta+\frac{\pi}{2}\leq\arg\left(z-e^{i\beta}\right)\leq\beta+\frac{3\pi}{2}$ to define the branch $\arg\left(z-e^{i\beta}\right)$, because $\beta+\frac{3\pi}{2}=\left(\beta+\frac{\pi}{2}\right)+\pi$.

Similarly, The branch $\arg\left(z-e^{i\alpha}\right)$ is defined by removing from $\mathbb{C}$ the cut $\left\{re^{i\alpha}\,\middle|\,r\geq 1\right\}$ and use the range $\alpha+\frac{\pi}{2}\leq\arg\left(z-e^{i\alpha}\right)\leq\alpha+\frac{3\pi}{2}$. Strictly speaking, we should also specify how to choose the value β for $e^{i\beta}$ and how to choose the value α for $e^{i\alpha}$. It matters when we are allowed to let α and β go through a range whose length is at least 2π. However, we are only interested in finding the Poisson integral kernel which, as we saw earlier, involves only the case of $\alpha=\theta$ and $\beta=\theta+d\theta$. So the length of the range for the values of α and β under consideration is limited to an extremely small number and certainly will be less than 2π. It is immaterial how we choose the value β for $e^{i\beta}$ and how we choose the value α for $e^{i\alpha}$.

When $z=e^{i\theta}$ is on the arc between $e^{i\beta}$ and $e^{i\alpha}$ in the counterclockwise sense so that $\beta<\theta<2\pi+\alpha$, we have

$$\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right)\equiv\frac{1}{2}\left(\beta-\alpha\right)$$

for the following reason. Take $e^{i\theta}$ very close to $e^{i\beta}$ but is just after $e^{i\beta}$ in the counterclockwise sense along the unit circle $\{|z|=1\}$. The value $\arg\left(e^{i\theta}-e^{i\beta}\right)$ is very close to that of $\arg\vec{BT_B}$. On the other hand, the value $\arg\left(e^{i\theta}-e^{i\alpha}\right)$ is very close to that of $\arg\vec{AB}$. Let $\hat{A}$ be a point on the line $\overline{AB}$ which is not on the same side of B as A so that when we go from A to B to $\hat{A}$ we are along the direction of the vector $\vec{AB}$. Then the constant value of

$$\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right)$$

should be equal to the angle $\angle\hat{A}BT_B$ which can be computed as follows. The angle $\angle AOB$ is equal to $\beta-\alpha$. The triangle ΔAOB is an isosceles triangle and the sum of its three angle $\angle AOB$, $\angle OBA$, $\angle OAB$ is π. Since the two angles $\angle OBA$ and $\angle OAB$ are equal, each one is equal to $\frac{\pi}{2} - \frac{1}{2}(\beta - \alpha)$. Let $\widehat{T_B}$ be a point on the line $\overline{BT_B}$ which is not on the same side of B as T_B so that when we go from $\widehat{T_B}$ to B to T_B we are along the direction of the vector $\vec{BT_B}$. The angle $\angle \hat{A}BT_B$ is the same as the angle $\angle\widehat{T_B}BA$. Since the sum of $\angle OBA$ and $\angle\widehat{T_B}BA$ is $\frac{\pi}{2}$, it follows from $\angle OBA = \frac{\pi}{2} - \frac{1}{2}(\beta - \alpha)$ that $\angle\widehat{T_B}BA = \frac{1}{2}(\beta - \alpha)$ and

$$\arg\left(\frac{z - e^{i\beta}}{z - e^{i\alpha}}\right) = \angle \hat{A}BT_B = \angle\widehat{T_B}BA = \frac{1}{2}(\beta - \alpha).$$

When $z = e^{i\theta}$ is on the arc between $e^{i\alpha}$ and $e^{i\beta}$ in the counterclockwise sense so that $\alpha < \theta < \beta$, we have

$$\arg\left(\frac{z - e^{i\beta}}{z - e^{i\alpha}}\right) = \pi + \frac{1}{2}(\beta - \alpha),$$

because the only difference for this case is that in the computation for the value for $\arg\left(z - e^{i\beta}\right)$ we have added π to it, as a result of the definition of the branch $\arg\left(z - e^{i\beta}\right)$.

From the above computation of the value for

$$\arg\left(\frac{z - e^{i\beta}}{z - e^{i\alpha}}\right)$$

we conclude that

$$\omega_{\alpha,\beta} = \frac{1}{\pi}\left(\arg\left(\frac{z - e^{i\beta}}{z - e^{i\alpha}}\right) - \frac{1}{2}(\beta - \alpha)\right),$$

as we can check, by using the value of

$$\arg\left(\frac{z - e^{i\beta}}{z - e^{i\alpha}}\right)$$

computed above in the two cases, that for $e^{i\theta}$ immediately after $e^{i\beta}$ in the counterclockwise sense the value of $\omega_{\alpha,\beta}$ is zero and that for $e^{i\theta}$ between $e^{i\alpha}$ and $e^{i\beta}$ in the counterclockwise sense the value of $\omega_{\alpha,\beta}$ is 1.

We would like to geometrically describe the Poisson kernel, using plane Euclidean geometry. Let P be the point z inside the open unit disk. Let A' be the intersection of the line joining A and P with the unit circle. Let B' be the intersection of the line joining B and P with the unit circle. We are going to express $\omega_{\alpha,\beta}$ in terms of the arc-length $\overset{\frown}{A'B'}$ from A' to B' around the unit circle. Since the exterior angle of a triangle equals the sum of the two non-adjacent interior angles, it follows that (when we consider the triangle $\Delta B'PA'$ and the exterior angle $\angle B'PA'$ at the vertex P)

$$\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right)=\angle B'PA'=\angle PB'A+\angle PAB'=\frac{1}{2}\overset{\frown}{AB}+\frac{1}{2}\overset{\frown}{A'B'}$$

Thus

$$\omega_{\alpha,\beta}=\frac{1}{\pi}\left(\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right)-\frac{1}{2}(\beta-\alpha)\right)$$

$$=\frac{1}{\pi}\left(\frac{1}{2}\overset{\frown}{AB}+\frac{1}{2}\overset{\frown}{A'B'}-\frac{1}{2}(\beta-\alpha)\right)=\frac{\overset{\frown}{A'B'}}{2\pi},$$

because $\overset{\frown}{AB}=\beta-\alpha$. Dividing this by $\beta-\alpha$ and taking limit, we get

$$(\natural)\qquad \lim_{\beta\to\alpha}\frac{\omega_{\alpha,\beta}}{\beta-\alpha}=\frac{1}{2\pi}\lim_{\beta\to\alpha}\frac{\overset{\frown}{A'B'}}{\beta-\alpha}.$$

We now derive the formula for the computation of $\overset{\frown}{A'B'}$ which would give us the Poisson integral formula. We denote the point A on the unit circle by ζ. Let P be a point z in the open unit disk and let w be the point A' which is the intersection of the line joining A and P with the unit circle. Since w, z, and ζ are collinear, there exists $\lambda\in\mathbb{R}$ such that $w-z=\lambda(\zeta-z)$. We have $w=\lambda(\zeta-z)+z$. Since $|w|^2=1$, it follows that

$$(\lambda(\zeta-z)+z)\,\overline{(\lambda(\zeta-z)+z)}=1,$$

which can be expanded into

$$\lambda^2|\zeta-z|^2+\lambda(\zeta-z)\bar{z}+z\lambda\left(\bar{\zeta}-\bar{z}\right)+|z|^2=1$$

or

$$(*)\qquad \lambda^2|\zeta-z|^2+\lambda\left(\zeta\bar{z}+z\bar{\zeta}-2r^2\right)+r^2=1.$$

From
$$|\zeta - z|^2 = (\zeta - z)\,\overline{(\zeta - z)} = |\zeta|^2 - \zeta\bar{z} - \bar{\zeta}z + |z|^2$$
it follows that
$$\zeta\bar{z} + \bar{\zeta}z = -\,|\zeta - z|^2 + |\zeta|^2 + |z|^2 = 1 + r^2 - |\zeta - z|^2\,,$$
which we put into $(*)$ and get

$$(\dagger) \qquad \lambda^2\,|\zeta - z|^2 + \lambda\left(1 - r^2 - |\zeta - z|^2\right) + r^2 = 1.$$

Rewrite $(\dagger)$ as
$$\lambda^2 + \lambda\left(\frac{1 - r^2}{|\zeta - z|^2} - 1\right) - \frac{1 - r^2}{|\zeta - z|^2} = 0,$$
which we can factor into
$$\left(\lambda + \frac{1 - r^2}{|\zeta - z|^2}\right)(\lambda - 1) = 0.$$
We have two solutions
$$\lambda = -\frac{1 - r^2}{|\zeta - z|^2}$$
and $\lambda = 1$. The solution $\lambda = 1$ just gives us back the intersection point A between the unit circle and the straight line joining A and P. Thus
$$\frac{|w - z|}{|\zeta - z|} = |\lambda| = \frac{1 - r^2}{|\zeta - z|^2}.$$
From the similar triangles $\Delta A'PB'$ and ΔBPA, we conclude that
$$\frac{\overline{A'B'}}{\overline{BA}} = \frac{\overline{A'P}}{\overline{AP}}$$
and
$$\lim_{B \to A} \frac{\widehat{A'B'}}{\widehat{AB}} = \lim_{B \to A} \frac{\overline{A'B'}}{\overline{BA}} = \lim_{B \to A} \frac{\overline{A'P}}{\overline{AP}}$$
which gives (due to $(\natural)$ and $\widehat{AB} = \beta - \alpha$) the formula
$$\lim_{\beta \to \alpha} \frac{\omega_{\alpha,\beta}}{\beta - \alpha} = \frac{1}{2\pi} \lim_{\beta \to \alpha} \frac{\widehat{A'B'}}{\beta - \alpha} = \frac{1}{2\pi} \lim_{B \to A} \frac{\widehat{A'B'}}{\widehat{AB}}$$

$$= \frac{1}{2\pi} \lim_{B \to A} \frac{\overline{A'P}}{\overline{AP}} = \frac{1}{2\pi} \frac{1 - r^2}{|\zeta - z|^2}.$$

This finishes the derivation of the Poisson integral formula

$$u(z) = \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1 - |z|^2}{|\zeta - z|^2} u(\zeta) \, d \left(\arg \zeta \right).$$