Power and Laurent Series Expansion, Classification of Isolated Singularities, and Computation of Residues.

We are going to take the next step into complex analysis from the practical viewpoint of the computation of definite integrals. We start out with the problem of computation of definite integrals of the form

$$\int_{\theta=0}^{2\pi} R\left(\cos\theta, \sin\theta\right) d\theta,$$

where $R\left(X, Y\right)$ is a rational function of two indeterminates X and Y over the complex number field. Recall that we used the following four steps.

(i) Use the parametrization $z = e^{i\theta}$, we get

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

to convert the given definite integral to a path integral over the unit circle of the form $\int_{|z|=1} S\left(z\right) dz$, where $S(z)$ is a rational function of the complex variable z.

(ii) Use partial fraction decomposition

$$S(z) = \sum_{j=1}^{\ell}\left(\sum_{\nu=1}^{k_j} \frac{A_{j,\nu}}{\left(z - a_j\right)^{\nu}}\right),$$

where $A_{j,\nu} \in \mathbb{C}$, $k_1, \cdots, k_\ell \in \mathbb{N}$, and $a_1, \cdots, a_\ell$ are distinct complex numbers.

(iii) Use special path integrals

$$\oint_{|z-a|=r} \left(z - a\right)^n dz = \begin{cases} 0 & \text{for } n \neq -1 \\ 2\pi i & \text{for } n = -1 \end{cases}$$

for $a \in \mathbb{C}$ and $r > 0$ and $n \in \mathbb{Z}$ by explicitly using the parametrization $\theta \mapsto z = a + re^{i\theta}$ for $0 \leq \theta \leq 2\pi$.

(iv) Finally use the Cauchy theorem for the smooth case and use small circles around those a_j's which are inside $\{|z|<1\}$ to compute the given definite integral as

$$\sum_{j\in J}\frac{2\pi i}{(k_j-1)!}\left(\frac{d^{k_1-1}}{dz^{k_j-1}}\left(\frac{(z-a_j)^{k_j}}{iz}R\left(\frac{1}{2}\left(z+\frac{1}{z}\right),\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\right)\right)_{z=a_j},$$

where J consists of all $1\leq j\leq \ell$ with $|a_j|<1$.

This procedure converts the computation of a definite integral to the computation of derivatives. For our next step into complex analysis from the practical viewpoint of the computation of definite integrals, we would like to be able to compute definite integrals for a more general class of integrands, also by converting the computation of definite integrals to the computations of derivatives. The key is to make this procedure work is to get the second step (ii) for functions more general than rational functions. Of course, these functions must be holomorphic (at least outside a finite set of points) so that we can apply the theorem of Cauchy-Goursat. The analog of the second step (ii) for general holomorphic functions is the following statement on power series expansion and its generalization to Laurent series expansion so that to each term in the convergent power or Laurent series we can apply the computation of special path integrals in (iii).

Power Series Expansion of a Holomorphic Function on a Disk. Let $0<R\leq\infty$ and $a\in\mathbb{C}$ and $f(z)$ be a holomorphic function on $\{|z-a|<R\}$. Then there exist $c_n\in\mathbb{C}$ for $n\in\mathbb{N}\cup\{0\}$ such that

$$f(z)=\sum_{n=0}^{\infty}c_n\,(z-a)^n$$

where the infinite series is absolute and uniform convergent on any compact subset of $\{|z-a|<R\}$. Moreover, $c_n\in\mathbb{C}$ is unique and is given by

$$c_n=\frac{1}{n!}f^{(n)}(a)=\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)dz}{(z-a)^{n+1}}$$

for any $0<r<R$.

Remarks. (1) The formula for c_n for the special case of $n = 0$ is the same as Cauchy's integral formula for the circle $\{|z - a| = R\}$ with the evaluation of the holomorphic function at the center a of the circle.

(2) An immediate consequence of the above statement on the power series expansion is that any holomorphic function $f(z)$ on $\{|z - a| < R\}$ is infinitely differentiable and, coupled with the theorem of Cauchy-Goursat, the above statement on the power series expansion gives the following

Cauchy's Integral Formula for Derivatives. Let D be a domain in $\mathbf{C}$ and $f(z)$ be a holomorphic function on D. Let C be a simple closed piecewise continuously differentiable curve in D such that the domain enclosed by C belongs to D. Suppose C goes around a point z of D precisely once in the counterclockwise sense. Then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

(3) Another immediate consequence of the above statement on the power series expansion is that, for any holomorphic function $f(z)$ which not identically zero, any of its zeroes is isolated, because at any zero a of $f(z)$ we can expand $f(z)$ in a convergent power series $\sum_{n=0}^{\infty} c_n (z - a)^n$ on $\{|z - a| < R\}$ for some $R > 0$ so that there is a smallest k with $c_k \neq 0$ and we can now write $f(z) = (z - a)^k g(z)$, where $g(z) = \sum_{n=k}^{\infty} c_n (z - a)^{n-k}$ is equal to $c_k \neq 0$ at $z = 0$ and is therefore nonzero on $\{|z - a| < R'\}$ for some $0 < R' < R$.

(4) The above statement of power series expansion will be obtained as a special case of the following statement on Laurent series expansion when each $c_n = 0$ for $n \leq -1$.

Laurent Series Expansion of a Holomorphic Function on an Annulus. Let $a \in \mathbb{C}$ and $0 \leq R_1 < R_2 \leq \infty$. Let $f(z)$ be holomorphic on the annulus $R_1 < |z - a| < R_2$. Then there exist $c_\nu \in \mathbb{C}$ for $\nu \in \mathbb{Z}$ such that

$$f(z) = \sum_{\nu=-\infty}^{\infty} c_\nu (z - a)^\nu$$

with absolute and uniform convergence on any compact subset of the annulus $R_1 < |z - a| < R_2$. Moreover, c_ν is unique and is given by

$$c_\nu = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - a)^{\nu+1}} d\zeta$$

for any circle C with radius r between R_1 and R_2 and center a.

Proof. Choose $R_1 < {R_1}' < {R_2}' < R_2$. Let C_1 be the circle of radius ${R_1}'$ centered at a and C_2 be the circle of radius ${R_2}'$ centered at a. For ${R_1}' < |z-a| < {R_2}'$ we have

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}dz$$

by Cauchy's integral formula. This means that $f(z)$ can be written as the limit of functions of the form $\sum_j \frac{A_j}{\zeta_j - z}$ and so any expansion of $\frac{1}{\zeta - z}$ as a function of z yields an expansion of $f(z)$. In particular, we can use the expansion of $\frac{1}{\zeta-z}$ as a geometric series. The expression $\frac{1}{z}$ (or $\frac{1}{\zeta-z}$) is known as the *Cauchy kernel*. The word "kernel" is in the sense of a "seed" for holomorphic functions, from which grow all holomorphic functions according to the Cauchy integral formula representing any holomorphic function on a disk as the limit of infinite $\mathbb{C}$-linear combination of translates of the kernel $\frac{1}{z}$. We now differentiate between two cases.

Case 1. $\zeta \in \partial {R_2}'$. Then

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{(\zeta - a)}\frac{1}{1 - (z-a)/(\zeta - a)} \\ &= \frac{1}{(\zeta - c)}\sum_{\nu=0}^{\infty}\left(\frac{z-a}{\zeta - c}\right)^{\nu} = \sum_{\nu=0}^{\infty}\frac{(z-a)^{\nu}}{(\zeta-c)^{\nu+1}}.\end{aligned}$$

Case 2. $\zeta \in \partial {R_1}'$. Then

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - a) - (z - a)} = \frac{-1}{(z - a)}\frac{1}{1 - (\zeta-a)/(z - a)} \\ &= \frac{-1}{(z - a)}\sum_{\mu=0}^{\infty}\left(\frac{\zeta-a}{z - a}\right)^{\mu} = -\sum_{\mu=0}^{\infty}\frac{(\zeta-a)^{\mu}}{(z-a)^{\mu+1}} \\ &= -\sum_{\nu=-1}^{-\infty}\frac{(z-a)^{\nu}}{(\zeta-a)^{\nu+1}} \text{ (after using } \nu = -(\mu+1)).\end{aligned}$$

By using these two expansions of $\frac{1}{\zeta-z}$, we get

$$(*) \qquad f(z) = \sum_{\nu=-\infty}^{\infty} c_{\nu}(z-a)^{\nu}$$

(known as the *Laurent series* of f on the $R_1 < |z-c| < R_2$) with

$$c_\nu = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - c)^{\nu+1}} d\zeta$$

for any circle C with radius r between $R_1{}'$ and $R_2{}'$ and center c. This Laurent series expansion (*) is convergent uniformly and absolutely on any compact subset of the annulus $R_1 < |z-c| < R_2$. (Note that for the special case when $f(z)$ is holomorphic on $|z-c| < R_2$ the theorem of Cauchy-Goursat yields $c_n = 0$ for $n \leq -1$ and the Laurent series becomes a power series so that the statement on power series is a special case for the statement on Laurent series. Moreover, for this special case, by term-by-term differentiation and then evaluating at $z = a$ we obtain $c_n = \frac{1}{n!} f^{(n)}(a)$ for $n \geq 0$.)

Suppose we have an expansion (*) which converges uniformly and absolutely on any compact subset of the annulus $R_1 < |z-c| < R_2$. Then by applying the operator

$$\int_C \frac{1}{(\zeta - c)^{n+1}} (\cdot) d\zeta$$

to both sides of (*) and using the fact that

$$\int_{C_{r,c}} (z-c)^n dz = 2\pi i$$

or 0 according as n is -1 or not, we conclude that c_n in the expansion (*) must be equal to

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - c)^{n+1}} d\zeta,$$

giving us the uniqueness of the coefficients c_n in a Laurent series. Q.E.D.

Isolated singularities. We now consider the special case $R_1 = 0$. In other words, a is an isolated singularity of the holomorphic function $f(z)$. The series

$$\sum_{\nu=-\infty}^{-1} c_\nu (z-a)^\nu$$

is known as the *principal part* of $f(z)$ at a. When the principal part

$$\sum_{\nu=-\infty}^{-1} c_\nu (z-a)^\nu$$

is zero, we say that we have a *removable singularity* at c. In the case of a removable singularity at a the function f near a is a convergent power series in $z-a$ and is therefore holomorphic at c. When the principal part

$$\sum_{\nu=-\infty}^{-1} c_\nu (z-a)^\nu$$

is nonzero and consists only of a finite number of nonzero terms, we say that we have a *pole* at a. The number k such that $c_{-k} \neq 0$ but $c_{-\ell} = 0$ for $\ell > k$ is called the *order* of the pole. When the principal part

$$\sum_{\nu=-\infty}^{-1} c_\nu (z-a)^\nu$$

contains an infinite number of nonzero terms, we say that we have an *essential singularity* at a.

Characterization of Removable Singularities. A removable singularity is characterized by the fact that f is L^2 near c. Suppose f is L^2 near a. We would like to verify that f is holomorphic at a. For $r > 0$, we have

$$\begin{aligned}\int_{\theta=0}^{2\pi} |f(re^{i\theta})|^2 d\theta &= \sum_{\mu,\nu=-\infty}^{\infty} \int_{\theta=0}^{2\pi} c_\mu \overline{c_\nu} r^{\mu+\nu} e^{i(\mu-\nu)\theta} d\theta \\ &= \sum_{\nu=-\infty}^{\infty} 2\pi\, c_\nu \overline{c_\nu} r^{2\nu} \geq 2\pi\, c_\nu \overline{c_\nu} r^{2\nu}.\end{aligned}$$

Suppose $c_\nu \neq 0$ for some $\nu \leq -1$. Then the finiteness of

$$\int_{r=0}^{\varepsilon} r dr \int_{\theta=0}^{2\pi} |f(re^{i\theta})|^2 d\theta$$

for some $\varepsilon > 0$ would imply the finiteness of $\int_{r=0}^{\varepsilon} 2\pi\, c_\nu \bar{c}_\nu^- r^{2\nu+1} dr$ which is not possible for any $\nu \leq -1$.

Characterization of Poles. Suppose a is a pole of order k. Then we can write $f(z) = (z-a)^{-k} g(z)$, where $g(z)$ is a convergent power series in $z-a$ near a and $g(a) \neq 0$. Thus $|f(z)|$ goes to infinity of order k as $z \to a$. Conversely, if $|f(z)|$ goes to infinity of order k as $z \to c$, then $(z-a)^k f(z)$ has a removable singularity at a and $(z-a)^k f(z)$ is equal to a convergent power series $g(z)$ in $z-a$ near a with $g(a) \neq 0$. Hence $f(z) = (z-a)^{-k} g(z)$ is a Laurent series with finite principal part and a is a pole of order k.

We call a function *meromorphic* if it is holomorphic except for isolated singularities which are poles.

Characterization of Essential Singularities. Suppose a is an essential singularity. Then the image under $f(z)$ of any deleted disk neighborhood of a is dense in $\mathbf{C}$. Suppose not. Then there exists some a such that $\frac{1}{f-a}$ is holomorphic on $0 < |z-a| < \varepsilon$ and is uniformly bounded there. Hence $\frac{1}{f-a}$ is holomorphic on $|z-a| < \varepsilon$. Let

$$\frac{1}{f-a} = \sum_{\nu=k}^{\infty} b_\nu (z-a)^\nu$$

be its power series expansion on $|z-a| < \varepsilon$ with $b_k \neq 0$. Then

$$f(z) = a + \frac{1}{(z-a)^k} \frac{1}{\sum_{\nu=0}^{\infty} b_{\nu+k}(z-a)^\nu}$$

and the power series expansion of

$$\frac{1}{\sum_{\nu=0}^{\infty} b_{\nu+k}(z-a)^\nu}$$

would make a a pole of order k for $f(z)$ which is a contradiction. Conversely when the image under $f(z)$ of any deleted disk neighborhood of a is dense in $\mathbf{C}$, a is an essential singularity of $f(z)$, because the other two cases of removable singularity and pole do not have this density property.

Residues. The *residue* of f at a is defined as c_{-1} which is by definition the integral

$$\frac{1}{2\pi i} \int_C f(\zeta) d\zeta$$

for any sufficiently small circle C centered at a. When a is a pole of order k, an alternative description of c_{-1} is

$$\frac{1}{(k-1)!} \left(\frac{d^{k-1}}{dz^{k-1}} (z-a)^k f(z) \right)_{z=c},$$

because of the power series expansion

$$(z-a)^k f(z) = \sum_{\nu=0}^{\infty} c_{\nu-k} (z-a)^k.$$

APPENDIX: Radius of Convergence of Power Series.

If the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges at some point $z_0 \neq 0$, then it converges uniformly and absolutely on $\{|z-a| \leq r|z_0-a|\}$ for any $0 < r < 1$. The reason is as follows. It follows from the convergence of the power series at $z_0 \neq a$ that $c_n (z_0-a)^n \to \infty$ as $n \to \infty$ and, in particular, $|c_n (z_0-a)^n| \leq M$ for some $M > 0$ independent of n. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} |c_n (z-a)^n| &= \sum_{n=0}^{\infty} |c_n (z_0-a)^n| \left|\frac{z_0-a}{z-a}\right|^n \\ &\leq M \sum_{n=0}^{\infty} r^n = \frac{M}{1-r} < \infty\end{aligned}$$

for $|z-a| \leq r|z_0-a|$. The supremum of $|z_0-a|$ taken over all z_0 such that the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges is called the *radius of convergence* of the power series.

Theorem (Hadamard). The radius R of convergence of the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ is given by

$$R = \liminf_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

Proof. Suppose $z_0 \in \mathbb{C}$ with

$$0 < |z_0-a| < \liminf_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

We are going to verify that the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges at z_0. Choose r such that

$$0 < |z_0-a| < r < \liminf_{n\to\infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

Then

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} < \frac{1}{r}.$$

There exists some $N \in \mathbb{N}$ such that

$$\sqrt[n]{|c_n|} \leq \frac{1}{r} \quad \text{for } n \geq N.$$

As a consequence there exists some positive number M such that $|c_n| r^n \leq M$ for all n. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} |c_n (z_0 - a)^n| &= \sum_{n=0}^{\infty} |c_n r^n| \left|\frac{z_0 - a}{r}\right|^n \\ &\leq M \sum_{n=0}^{\infty} \left|\frac{z_0 - a}{r}\right|^n = \frac{M}{1 - \left|\frac{z_0 - a}{r}\right|} < \infty\end{aligned}$$

and the power series $\sum_{n=0}^{\infty} c_n (z - a)^n$ converges at z_0.

Conversely we now assume that the power series $\sum_{n=0}^{\infty} c_n (z - a)^n$ converges at some complex number $z_0 \neq a$. We would like to show that

$$|z_0 - a| \leq \liminf_{n \to \infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

The convergence of $\sum_{n=0}^{\infty} c_n (z_0 - a)^n$ implies that $c_n (z_0 - a)^n \to \infty$ as $n \to \infty$ and, in particular, $|c_n (z_0 - a)^n| \leq M$ for some $M > 0$ independent of n. Thus

$$\sqrt[n]{|c_n|} \leq \frac{\sqrt[n]{M}}{|z_0 - a|} \quad \text{for all } n \in \mathbb{N}$$

and

$$\limsup_{n \to \infty} \sqrt[n]{|c_n|} \leq \frac{\lim_{n \to \infty} \sqrt[n]{M}}{|z_0 - a|} = \frac{1}{|z_0 - a|}$$

which implies

$$|z_0 - a| \leq \liminf_{n \to \infty} \frac{1}{\sqrt[n]{|c_n|}}.$$

Q.E.D.

Theorem (Ratio Test for Convergence of Power Series. Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n (z - a)^n$. Suppose there exists $N_0 \in \mathbb{N}$ such that $c_n \neq 0$ for $n \geq N_0$. Then

(a) $\displaystyle \liminf_{n \to \infty} \frac{|c_n|}{|c_{n+1}|} \leq R.$

(b) $\displaystyle R \leq \limsup_{n \to \infty} \frac{|c_n|}{|c_{n+1}|}.$

Proof. For (a) we assume that

$$(**) \qquad R < \liminf_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}$$

and show that it leads to a contradiction. From $(**)$ we can choose $z_0 \in \mathbb{C}$ with

$$R < |z_0 - a| < \liminf_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}$$

and we are going to show the power series $\sum_{n=0}^{\infty} c_n (z_0 - a)^n$ converges, which would contradict $R < |z_0 - a|$. Take $r > 0$ with

$$|z_0 - a| < r < \liminf_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}.$$

Then for some $N \in \mathbb{N}$ with $N \geq N_0$ we have

$$r \leq \frac{|c_n|}{|c_{n+1}|} \quad \text{for } n \geq N.$$

Since $c_n \neq 0$ for $n \geq N$, by induction on $n \geq N$ we conclude that

$$|c_n r^n| \leq |c_N| \, r^N \quad \text{for } n \geq N$$

and, as a consequence,

$$\sum_{n=N}^{\infty} |c_n (z_0 - a)^n| = \sum_{n=N}^{\infty} |c_n r^n| \left(\frac{|z_0 - a|}{r}\right)^n$$

$$\leq |c_N| \, r^N \sum_{n=N}^{\infty} \left(\frac{|z_0 - a|}{r}\right)^n = \frac{|c_N| \, r^N}{1 - \frac{|z_0 - a|}{r}} < \infty.$$

For the verification of (b) we assume that

$$(\dagger\dagger) \qquad \liminf_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} < R$$

and show that it leads to a contradiction. From $(\dagger\dagger)$ we can choose $z_0 \in \mathbb{C}$ with

$$\liminf_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} < |z_0 - a| < R$$

and we are going to show the power series $\sum_{n=0}^{\infty} c_n (z_0 - a)^n$ does not converge, which would contradict $|z_0 - a| < R$. Take $r > 0$ with

$$\limsup_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} < r < |z_0 - a| \, .$$

Then there exists a subsequence $\{n_\nu\}_{\nu\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$\frac{|c_{n_\nu}|}{\left|c_{n_{\nu+1}}\right|} \geq r \quad \text{for } n \geq N.$$

Then for some $N \in \mathbb{N}$ with $N \geq N_0$ we have

$$\frac{|c_n|}{|c_{n+1}|} \geq r \quad \text{for } n \geq N.$$

Since $c_n \neq 0$ for $n \geq N$, by induction on $n \geq N$ we conclude that

$$|c_n r^n| \geq |c_N| \, r^N \quad \text{for } n \geq N$$

and

$$|c_n (z_0 - a)^n| \geq |c_N| \, r^N > 0 \quad \text{for } n \geq N.$$

In particular, the sequence $c_n (z_0 - a)^n$ cannot approach 0 as $n \to \infty$. As a consequence, the power series $\sum_{n=0}^{\infty} c_n (z_0 - a)^n$ does not converge, which would contradict $|z_0 - a| < R$. Q.E.D.