Schwarz-Christoffel Transformations

Schwarz-Christoffel transformations are used to map the upper half-plane $H := \{y > 0\}$ to a given polygon which may be bounded or unbounded. An unbounded polygon simply means a domain in $\mathbb{C}$ defined by a finite number of line-segments and rays. Let us start out with an n-sided polygon Ω with vertices $w_1, \cdots, w_{n-1}$. We do not put down the n-th vertex w_n to allow the possibility that w_n is ∞, in which case the n-sided polygon Ω is unbounded. We would like to study the problem by constructing a holomorphic map $w = f(z)$ from H to Ω so that $n-1$ real points $x_1 < x_2 < \cdots < x_{n-1}$ are mapped respectively to the prescribed $n-1$ vertices $w_1, w_2, \cdots, w_{n-1}$ of the n-sided polygon Ω. The important requirement is that the holomorphic map $w = f(z)$ should map the straight line segment $[x_j, x_{j+1}] \subset \mathbb{R}$ to the line-segment in $\mathbb{C}$ joining w_j to w_{j+1}. Let us recall how a holomorphic map transforms the direction of the tangent to a curve in $\mathbb{C}$.

We now recall the mapping behavior of the holomorphic map $w = f(z)$ at a point z_0 where its derivative $f'(z_0)$ is nonzero. Take a smooth curve $t \mapsto z(t)$ passing through the point z_0 so that $z(0) = z_0$. The image of the curve $t \mapsto w(t) := f(z(t))$ passes through the point $w_0 := f(z_0)$. By the chain rule and the Cauchy-Riemann equation we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + i\frac{\partial f}{\partial x}\frac{dy}{dt} \\ &= \frac{\partial f}{\partial x}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) \\ &= \frac{\partial f}{\partial x}\frac{dz}{dt} = f'(z)\frac{dz}{dt}.\end{aligned}$$

This means that the angle made between the tangent to the curve $t \mapsto z(t)$ at z_0 and the real axis in the z variable is equal to the angle made between the tangent to the curve $t \mapsto w(t)$ at w_0 and the real axis in the w variable plus the angle of the polar representation of $f'(z_0)$. So, if the angle of $f'(z)$ (which we will from now on call the argument of $f'(z)$ and denote by $\arg f'(z)$) stays unchanged as z varies along $[x_j, x_{j+1}]$ from x_j to x_{j+1}, then the line $[x_j, x_{j+1}]$ will be mapped to part of a straight line, because the curve (x_j, x_{j+1}) is mapped to its image curve C in $\mathbb{C}$ so that the tangent direction at every point of C is the same as the tangent direction of $(x_j,\, x_{j+1})$ plus the constant value of $\arg f'(z)$ for $z \in (x_j,\, x_{j+1})$.

When z goes along $\mathbb{R}$ from the left of x_j to the right of x_{j+1}, we want the image to turn an angle of magnitude θ_j when it follows the sides of the n-sided polygon going through the vertex w_j in the counterclockwise sense. To achieve this requirement, we would have to require that the constant value of $\arg f'(z)$ at z on $\mathbb{R}$ to the left of x_j should increase by the amount θ_j to get to the constant value of $\arg f'(z)$ at z on $\mathbb{R}$ to the right of x_j.

We now look for a holomorphic function $g(z)$ to serve as $f'(z)$ which would have this property of being constant along $\mathbb{R}$ to the left of x_j and then increase by the amount θ_j to a constant value along $\mathbb{R}$ to the right of x_j. When $\theta_j = -\pi$, one simple function to fit the requirement is $g(z) = z - x_j$ which satisfies the property that $\arg g(z) = \arg(z - x_j) \equiv -\pi$ for $z \in \mathbb{R}$ and $z < x_j$ and $\arg g(z) = \arg(z - x_j) \equiv 0$ for $z \in \mathbb{R}$ and $z > x_j$. To assign a numerical value to the angle $\arg g(z) = \arg(z - x_j)$ would require a choice of the range of the values, which is the same as choosing one branch of the function $\log(z - x_j)$ and then taking its imaginary part $\arg(z - x_j)$. When we say that $\arg g(z) = \arg(z - x_j) \equiv -\pi$ for $z \in \mathbb{R}$ and $z < x_j$ and $\arg g(z) = \arg(z - x_j) \equiv 0$ for $z \in \mathbb{R}$ and $z > x_j$, we implicitly make the choice of choosing the cut $\{x = x_j,\, y \leq 0\}$ to define a branch of $\log(z - x_j)$. That particular cut is chosen to define $\log(z - x_j)$, because we would like the function $\log(z - x_j)$ to be defined on the upper half-plane H.

We now consider the general value $k_j\pi$ for the amount θ_j of jump instead of the special value $\theta_j = -\pi$. This we can do by multiplying $\arg(z - x_j)$ by the constant $-k_j$, which is the same as using $(z - x_j)^{-k_j}$ instead of $z - x_j$. This takes care of one single jump of $\arg g(z)$ by the amount $k_j\pi$ at the point $z = x_j$. How about all the other jumps at all the other real points among $x_1, x_2, \cdots, x_{n-1}$? An easy solution is to add the individual jumps together. When the jump of $\arg g(z)$ is $k_j\pi$ at $z = x_j$, we should use just take

$$\arg g(z) = \sum_{j=1}^{n-1} (-k_j) \arg(z - x_j)\,.$$

In other words, we should just choose

$$g(z) = \prod_{j=1}^{n-1} (z - x_j)^{-k_j}\,.$$

Recall that $g(z)$ is actually a stand-in for $f'(z)$. So we should use

$$f'(z) = \prod_{j=1}^{n-1} (z - x_j)^{-k_j}$$

in order for the image $w = f(z)$ to achieve the correct turning of corners along the n-sided polygon Ω at the vertices $w_1, \cdots, w_{n-1}$ in the counterclockwise sense.

We should do more than just achieving the correct turning of corners along the n-sided polygon. We should make sure that the ray $(-\infty, x_1]$ is mapped to the side of Ω just before w_1 in the counterclockwise sense. In order to get the initial matching of $(-\infty, x_1]$ with the side of Ω just before w_1, we can multiply $f'(z)$ by a nonzero complex constant A, resulting in the entire image of $f(z)$ being rotated by the argument $\arg A$ of A. So far as matching the orientation is concerned, the absolute value $|A|$ of the complex number A plays no rôle. The absolute value $|A|$ of the complex number A, however, plays the rôle of magnification. Since only the derivative $f'(z)$ is specified, to get back to the holomorphic function $f(z)$ there is the question of the constant of integration. If we use a definite integral with x_1 as the lower limit, we should use w_1 as the constant of integration, because we would like x_1 to be mapped to w_2. Thus we have the following formula for the Schwarz-Christoffel transformation

$$f(z) = A \int_{z_0}^{z} \prod_{j=1}^{n-1} (\zeta - x_j)^{-k_j} \, d\zeta + B,$$

where, for example, z_0 and B can be chosen respectively to be x_1 and w_1.

So far we have been talking about fitting the angles and one initial direction and one initial point. How about fitting all the vertices so that the n vertices $w_1, w_2, \cdots, w_{n-1}, w_n$ (with w_n possibly equal to ∞) are precisely the images of $x_1, x_2, \cdots, x_{n-1}, \infty$ under $w = f(z)$ respectively? Of course, once all the angles fit together with one initial direction and one initial point, the only remaining problem is the lengths of the sides. There are precisely n real numbers (including the possibility of ∞) for the lengths of the sides of the n-sided polygon Ω. On the other hand, we have precisely n real degrees of freedom, namely $|A|$, $x_1, \cdots, x_{n-1}$, to do the job.

One simple way of getting the value for B is to choose $z_0 = x_1$. Then $B = w_1$. Since the line segment $[x_1, x_2]$ is mapped to the side $[w_1, w_2]$, of the n-sided polygon Ω, for $1 \leq \nu \leq n-2$ we have the $n-2$ equations

$$(*)_\nu \qquad A \int_{t=x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} (t - x_j)^{-k_j}\, d\zeta = w_{\nu+1} - w_\nu.$$

For $t \in \mathbb{R}$ the complex number $(t - x_j)^{-k_j}$ is given by

$$(\ddagger) \qquad (t - x_j)^{-k_j} = \begin{cases} |t - x_j| & \text{for } \ t > x_j \\ \\ |t - x_j|\, e^{-ik_j\pi} & \text{for } \ t < x_j \end{cases}$$

according to the choice of the branch for the function $(\zeta - x_j)^{-k_j}$ defined for $\operatorname{Im} \zeta \geq 0$. The equation $(*)_\nu$ can now be rewritten as

$$A \left(\prod_{j=\nu+1}^{n-1} e^{-ik_j\pi} \right) \int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} |t - x_j|^{-k_j}\, dt = w_{\nu+1} - w_\nu.$$

This gives right away the value for $\arg A$, for example, from the equation $(*)_1$, namely

$$\arg A = \arg (w_2 - w_1) + \sum_{j=2}^{n-1} k_j \pi.$$

At this point we are left with the n unknowns $|A|, x_1, \cdots, x_{n-1}$. By taking the absolute values of both sides of $(*)_\nu$, we get

$$(\dagger)_\nu \qquad \int_{x_\nu}^{x_{\nu+1}} \prod_{j=1}^{n-1} |t - x_j|^{-k_j}\, dt = \frac{|w_{\nu+1} - w_\nu|}{|A|}$$

for $1 \leq \nu \leq n-1$. Since we have n unknowns $|A|, x_1, \cdots, x_{n-1}$ to determine, the $n-2$ equations $(*)_1, \cdots, (*)_{n-2}$ are not enough. We still need two more equations. One piece of information we have not yet used, namely the vertex w_n to worry about. This vertex w_n should be reached by the limit $f(t)$ both

(i) by letting $x_{n-1} < t < \infty$ go to $+\infty$ and

(i) by letting $-\infty < t < x_1$ go to $-\infty$.

So we have the two equations

$$A\int_{t=x_{n-1}}^{+\infty}\prod_{j=1}^{n-1}(t-x_j)^{-k_j}\,dt=w_n-w_{n-1},$$

$$A\int_{t=-\infty}^{x_1}\prod_{j=1}^{n-1}(t-x_j)^{-k_j}\,dt=w_1-w_n.$$

Using (‡), we can rewrite these two equations as

$$A\int_{t=x_{n-1}}^{+\infty}\prod_{j=1}^{n-1}|t-x_j|^{-k_j}\,dt=w_n-w_{n-1},$$

$$A\left(\prod_{j=1}^{n-1}e^{-ik_j\pi}\right)\int_{t=-\infty}^{x_1}\prod_{j=1}^{n-1}(t-x_j)^{-k_j}\,dt=w_1-w_n.$$

Again we can take the absolute value of both sides of the two equations and get

$$(\dagger)_{n-1}\qquad\int_{t=x_{n-1}}^{+\infty}\prod_{j=1}^{n-1}|t-x_j|^{-k_j}\,dt=\frac{|w_n-w_{n-1}|}{|A|},$$

$$(\dagger)_{n}\qquad\int_{t=-\infty}^{x_1}\prod_{j=1}^{n-1}|t-x_j|^{-k_j}\,dt=\frac{|w_1-w_n|}{|A|}.$$

We now use the n equations $(\dagger)_\nu$ for $1\leq\nu\leq n$ to solve for the n unknowns $|A|, x_1,\cdots,x_n$. These n equations are integral equations with the $n-1$ unknowns $x_1,\cdots,x_{n-1}$ both in the integrands as well as in the upper limits of the integrals. In general, it is very difficult to solve these n equations simultaneously.

Example of a Conformal Mapping from the Upper Half Plane to an Equilateral Triangle. Given an equilateral triangle Ω with $w_1=0$ and $w_2>0$ and $\operatorname{Im} w_3>0$. We seek a conformal map from the upper half-plane $H:=\{\operatorname{Im} z>0\}$ to Ω with $x_1=-1$ and $x_2=1$. We do not specify

$|w_2 - w_1| = |w_2|$ but instead normalize $A = 1$. The conformal map $w = f(z)$ will be given by the Schwarz-Christoffel transformation

$$f(z) = \int_{-1}^{z} \frac{d\zeta}{(\zeta+1)^{\frac{2}{3}} (\zeta-1)^{\frac{2}{3}}}.$$

We would like to determine the length of a side $|w_2|$ of the equilateral triangle Ω so that we know precisely what the equilateral triangle Ω with the normalization $|A| = 1$. In order for the Schwarz-Christoffel transformation to map 1 to $w_2 > 0$, according to the preceding discussion we must have $\arg A = \frac{2\pi}{3}$. We have the equation

$$\int_{-1}^{1} \frac{dt}{(t+1)^{\frac{2}{3}} (t-1)^{\frac{2}{3}}} = w_2.$$

From the above discussion on the values of the chosen branches of the factors in the integrand, we have

$$e^{i\frac{2\pi}{3}} \int_{-1}^{1} \frac{dt}{|t+1|^{\frac{2}{3}} |t-1|^{\frac{2}{3}}} = w_2.$$

By taking the absolute values of both sides of the equation, we get

$$\int_{-1}^{1} \frac{dt}{|1+t|^{\frac{2}{3}} |1-t|^{\frac{2}{3}}} = |w_2|.$$

We rewrite the equation as

$$\int_{-1}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}} = w_2,$$

because $w_2 > 0$. To evaluate the definite integral

$$\int_{-1}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}} = 2\int_{0}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}},$$

we use the transformation $\tau = t^2$ and get $d\tau = 2tdt = 2\tau^{\frac{1}{2}}dt$ and

$$2\int_{0}^{1} \frac{dt}{(1-t^2)^{\frac{2}{3}}} = \int_{0}^{1} \frac{d\tau}{\tau^{\frac{1}{2}} (1-\tau)^{\frac{2}{3}}}$$

which is equal to the value $B\left(\frac{1}{2},\frac{1}{3}\right)$ of the beta function

$$B(x,y)=\int_0^1 \tau^{x-1}(1-\tau)^{y-1}\,d\tau.$$

Our final conclusion of this example is that the holomorphic map

$$z\mapsto\ e^{i\frac{2\pi}{3}}\int_{-1}^{z}\frac{d\zeta}{(\zeta+1)^{\frac{2}{3}}(\zeta-1)^{\frac{2}{3}}}$$

maps the upper half-plane to the equilateral triangle in the upper half-plane whose base is $\left[0,B\left(\frac{1}{2},\frac{1}{3}\right)\right]$ and is the image of $[-1,1]$ with the sense preserved.

Example of Using a Scharz-Christoffel Transformation to Show that the Sine Function Maps a Vertical Upper Half-Strip to the Upper Half Plane. Let our 3-sided polygon Ω to be the vertical upper half-strip $\left\{-\frac{\pi}{2}<x<\frac{\pi}{2},\,y>0\right\}$ with the two vertices $w_1=-\frac{\pi}{2}$ and $w_2=\frac{\pi}{2}$. We would like to write down the Schwarz-Christoffel transformation to map the upper half-plane $H:=\{\operatorname{Im}z>0\}$ to Ω with $x_1=-1$ and $x_2=1$. The two angles at w_1 and w_2 are both $\frac{\pi}{2}$. The Schwarz-Christoffel transformation $w=f(z)$ is

$$f(z)=-\frac{\pi}{2}+A\int_{-1}^{z}\frac{d\zeta}{(\zeta+1)^{\frac{1}{2}}(\zeta-1)^{\frac{1}{2}}}.$$

for some $A>0$, because $w_1=-\frac{\pi}{2}$. To determine A, we put in the value $z=1$ to get

$$\frac{\pi}{2}=-\frac{\pi}{2}+A\int_{-1}^{1}\frac{d\zeta}{(\zeta+1)^{\frac{1}{2}}(\zeta-1)^{\frac{1}{2}}}.$$

because $w_2=\frac{\pi}{2}$. From the above discussion on the values of the chosen branches of the factors in the integrand, after we move $-\frac{\pi}{2}$ from the left-hand side of the equation to the right-hand, we get

$$\pi=A\int_{-1}^{1}\frac{i\,dt}{|1+t|^{\frac{1}{2}}\,|1-t|^{\frac{1}{2}}}.$$

We know that

$$\int_{-1}^{1}\frac{dt}{|1+t|^{\frac{1}{2}}\,|1-t|^{\frac{1}{2}}}=\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=\sin^{-1}t\,\Big|_{t=-1}^{t=1}=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi.$$

Hence $A = -i$. Now our Schwarz-Christoffel transformation is

$$f(z) = -\frac{\pi}{2} + A \int_{-1}^{z} \frac{d\zeta}{(\zeta+1)^{\frac{1}{2}} (\zeta-1)^{\frac{1}{2}}}.$$

For $z = t$ in the interval $(-1, 1)$, according to the values of the chosen branches of the factors in the integrand we have

$$f(t) = -\frac{\pi}{2} + A \int_{-1}^{t} \frac{i\,ds}{\sqrt{1-s^2}} = -\frac{\pi}{2} + \int_{-1}^{t} \frac{ds}{\sqrt{1-s^2}} = \int_{0}^{t} \frac{ds}{\sqrt{1-s^2}} = \sin^{-1} t,$$

because

$$\int_{-1}^{0} \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{2}.$$

The range of $\sin^{-1} t$ for $-1 < t <$ would have to be the range of $f(t)$ for $-1 < t <$ and hence must be $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Taking the inverse of $\sin^{-1} z$, we conclude that the holomorphic map $z = \sin w$ maps the vertical upper half-strip

$$\left\{ w = u + iv \,\middle|\, -\frac{\pi}{2} < u < \frac{\pi}{2},\ v > 0 \right\}$$

onto the upper half-plane $\{y > 0\}$ with $w = -\frac{\pi}{2}$ corresponding to $z = -1$ and $w = \frac{\pi}{2}$ corresponding to $z = 1$.