Legendre's Necessary Condition for Local Minimum

We now discuss second variation to investigate whether an extremal function gives a local minimum or not. We consider the variation problem in the simplest context which is to minimize

$$J=\int_a^b F\left(x,y,y'\right)dx$$

subject to $y(a)=A$ and $y(b)=B$. Assume that we have an extremal $y=y(x)$. To investigate whether it is a local minimum, we put it in a 1-parameter family $y=y(x,t)$ with the given extremal corresponding to $t=0$. The functional J evaluated at each of the function $y=y(x,t)$ in the family gives a function $J[t]$ in t. Let $h=\frac{\partial}{\partial t}y(x,t)\Big|_{t=0}$. We consider the Taylor series expansion of $J[t]$ centered at $t=0$ (after using the chain rule and applying integration by parts to the linear term) and get

$$\begin{aligned}J[t]=J[0]&+\int_a^b\left(F_y-\frac{d}{dx}F_{y'}\right)hdx\\&+\frac{1}{2}\int_a^b\left(F_{yy}h^2+2F_{yy'}hh'+F_{y'y'}h'^2\right)dx+O(t^3),\end{aligned}$$

where $O\left(\cdot\right)$ is the Landau symbol. Since $y=y(x,t)$ is an extremal function when $t=0$, the Euler-Lagrange equation holds at $t=0$ (*i.e.,* the linear part vanishes at $t=0$) and we get

$$J[t]=J[0]+\frac{1}{2}\int_a^b\left(F_{yy}h^2+2F_{yy'}hh'+F_{y'y'}h'^2\right)dx+O(t^3).$$

For the extremal to be a local minimum, we cannot have

$$\frac{1}{2}\int_a^b\left(F_{yy}h^2+2F_{yy'}hh'+F_{y'y'}h'^2\right)dx<0$$

for some h with $h(a)=h(b)=0$, because along the direction of that particular h in the function space we will get a local maximum. So we want

$$\int_a^b\left(F_{yy}h^2+2F_{yy'}hh'+F_{y'y'}h'^2\right)dx\geq 0$$

for all h with $h(a) = h(b) = 0$. If the coefficients F_{yy} and $F_{y'y'}$ are nonnegative, the two terms

$$F_{yy}h^2 + F_{y'y'}h'^2$$

of the integrand are nonnegative. The situation is less clear with the middle term

$$2F_{yy'}hh'$$

of the integrand. We would like to perform an integration by parts to convert it to a form under better control, like the situation of a term with the variable occurring in a perfect square. The key is to use $(h^2)' = 2hh'$. Applying this, we get

$$2F_{yy'}hh' = \frac{d}{dx}\left(F_{yy'}h^2\right) - \left(\frac{d}{dx}F_{yy'}\right)h^2.$$

Thus

$$\int_a^b \left(F_{yy}h^2 + 2F_{yy'}hh' + F_{y'y'}h'^2\right)dx = \int_a^b \left(\left(F_{yy} - \left(\frac{d}{dx}F_{yy'}\right)\right)h^2 + F_{y'y'}h'^2\right).$$

To simplify the notations, we let

$$P(x) = F_{y'y'},$$
$$Q(x) = F_{yy} - \left(\frac{d}{dx}F_{yy'}\right)$$

evaluated along the extremal $y = y(x, 0)$, so that the problem is reduced to the study of the condition

$$\int_a^b \left(P(x)h'(x)^2 + Q(x)h(x)^2\right)dx \geq 0$$

for all $h(x)$ with $h(a) = h(b) = 0$.

Legendre's Necessary Condition. A necessary condition for $y = y(x)$ to be a local minimum is that $P(x) \geq 0$.

Proof. Suppose the contrary so that, for some α, $\beta > 0$ and some $x_0 \in (a, b)$, the function $P(x) \leq -\beta$ on $[x_0 - \alpha, x_0 + \alpha]$. Let M be a positive number which is no less than the supremum of $|Q(x)|$ on $[a, b]$. We are going to derive a contradiction by choosing some $h(x)$ with $h(a) = h(b) = 0$ so that

$$\int_a^b \left(P(x)h'(x)^2 + Q(x)h(x)^2\right)dx < 0.$$

The function $h(x)$ which we choose is equal to

$$\sin^2 \frac{\pi\,(x-x_0)}{\alpha} \quad \text{for} \quad x \in [x_0-\alpha, x_0+\alpha]$$

and is 0 otherwise. Then $h'(x)$ is equal to

$$\frac{\pi}{\alpha}\, 2\sin\frac{\pi\,(x-x_0)}{\alpha}\cos\frac{\pi\,(x-x_0)}{\alpha} = \frac{\pi}{\alpha}\,\sin\frac{2\pi\,(x-x_0)}{\alpha}$$

for $x \in [x_0-\alpha, x_0+\alpha]$ and is 0 otherwise. Thus

$$\begin{aligned} &\int_a^b \left(P(x)h'(x)^2 + Q(x)h(x)^2\right) dx \\ &= \int_{x_0-\alpha}^{x_0+\alpha} \left(P(x)\frac{\pi^2}{\alpha^2}\sin^2\frac{2\pi\,(x-x_0)}{\alpha} + Q(x)\sin^4\frac{\pi\,(x-x_0)}{\alpha}\right) dx \\ &\leq -\beta\frac{\pi^2}{\alpha^2}\int_{x_0-\alpha}^{x_0+\alpha}\sin^2\frac{2\pi\,(x-x_0)}{\alpha}dx + 2M\alpha \\ &= -\beta\frac{\pi^2}{\alpha^2}\int_{x_0-\alpha}^{x_0+\alpha}\frac{1}{2}\left(1-\cos\frac{4\pi\,(x-x_0)}{\alpha}\right)dx + 2M\alpha \\ &= -\beta\frac{\pi^2}{\alpha} + 2M\alpha \end{aligned}$$

which is < 0 when $\alpha^2 < \frac{\beta\pi^2}{M}$. This is a contradiction. Q.E.D.