Conjugate Points and Sufficient Condition for Local Minimum

We now discuss a sufficient condition for an extremal function to be a local minimum by using conjugate points. We consider the variation problem in the simplest context which is to minimize

$$J=\int_a^b F\left(x,y,y'\right)dx$$

subject to $y(a)=A$ and $y(b)=B$. Assume that we have an extremal $y=y(x)$. Let

$$P(x)=F_{y'y'},$$
$$Q(x)=F_{yy}-\left(\frac{d}{dx}F_{yy'}\right)$$

evaluated along the given extremal $y=y(x)$. To investigate whether it is a local minimum, we put it in a 1-parameter family $y=y(x,t)$ with the given extremal corresponding to $t=0$. The functional J evaluated at each of the function $y=y(x,t)$ in the family gives a function $J[t]$ in t. Let $h=\frac{\partial}{\partial t}y(x,t)\Big|_{t=0}$. As derived earlier, the Taylor series expansion of $J[t]$ centered at $t=0$ is

$$J[t]=J[0]+\frac{1}{2}\int_a^b\left(P(x)h'(x)^2+Q(x)h(x)^2\right)dx+O\left(t^3\right)$$

for all $h(x)$ with $h(a)=h(b)=0$. We will derive a sufficient condition so that

$$\int_a^b\left(P(x)h'(x)^2+Q(x)h(x)^2\right)dx>0$$

for all $h(x)$ with $h(a)=h(b)=0$. The idea is to apply the method of the calculus of variations to the functional

$$\int_a^b\left(P(x)h'(x)^2+Q(x)h(x)^2\right)dx,$$

where the dependent variable is $h(x)$ subject to the boundary condition $h(a)=h(b)=0$. The Euler-Lagrange equation is

$$(\dagger)\qquad -\frac{d}{dx}\left(Ph'\right)+Qh=0.$$

There is another interpretation of the equation (†) given as follows. Assume that our extremal $y = y(x)$ is one function among a 1-parameter family of functions $y = y(x,t)$ so that for each t the function $x \mapsto y(x,t)$ as a function of x satisfies the Euler-Lagrange equation

$$(*) \qquad F_y - \frac{d}{dx} F_{y'} = 0.$$

We now let $h = \frac{\partial}{\partial t} y(x,t)\Big|_{t=0}$ and differentiate the equation $(*)$ with respect to t and set $t = 0$ to get

$$F_{yy} h + F_{yy'} h' - \frac{d}{dx}\left(F_{y'y} h + F_{y'y'} h'\right) = 0$$

which after expansion and cancelation becomes

$$\left(F_{yy} - \frac{d}{dx} F_{y'y}\right) h - \frac{d}{dx}\left(F_{y'y'} h'\right) = 0$$

which is the equation (†). In this interpretation the equation (†) is the equation for the variation of functions satisfying the Euler-Lagrange equation.

A motivating example for the investigation of a sufficient condition for a local minimum is the situation of a great-circle arc on a unit sphere. If the length of the great-circle arc is $> \pi$, the great-circle arc is not a local minimum. The condition of the the length of the great-circle arc is $> \pi$ can be reformulated as the arc containing two antipodal points (for example, the north pole and the south pole). When two points P and Q on the unit sphere are antipodal, a 1-parameter family of great circles pass through both points P and Q. The variation h of the functions defining the great circle would satisfy the equation (†) with $h(P) = h(Q) = 0$. The notion of the antipodal points P and Q can be formulated for the problem of minimizing

$$J = \int_a^b F\left(x, y, y'\right) dx$$

subject to $y(a) = A$ and $y(b) = B$. In the general setting the analogs of the antipodal points are called *conjugate points*.

Definition (Conjugate Points). A point $\tilde{a} \in (a, b]$ is called a conjugate point of a if there exists some solution $h(x) \not\equiv 0$ on $[a, \tilde{a}]$ of

$$-\frac{d}{dx}\left(P h'\right) + Q h = 0$$

such that $h(a) = h\left(\tilde{a}\right) = 0$.

The key idea of getting a sufficient condition is to add an exact differential (whose integral over $[a, b]$ vanishes) to

$$\int_a^b \left(P(x)h'(x)^2 + Q(x)h(x)^2\right) dx$$

to make it an integral of a perfect square. The Legendre's necessary condition for a local minimum is that $P(x) \geq 0$. It is reasonable to assume $P(x) > 0$ on $[a, b]$ in our study of a sufficient condition for a local minimum. As the exact differential we are going to add to the integral we choose $d\left(wh^2\right)$ with $w = w(x)$ to be determined (and use the vanishing of h at the end-points). We have

$$Ph'^2 + Qh^2 + \frac{d}{dx}\left(wh^2\right) = Ph'^2 + 2whh' + \left(Q + w'\right)h^2.$$

This will become the perfect square

$$P\left(h' + \frac{w}{P}\,h\right)^2$$

if

$$(\natural) \qquad P\left(Q + w'\right) = w^2.$$

The integral

$$\int_a^b P\left(h' + \frac{w}{P}\,h\right)^2 dx$$

is nonnegative and, if it is zero, the first-order differential equation

$$h' + \frac{w}{P}\,h = 0$$

must be satisfied, which together with the initial condition $h(a) = 0$ would force $h \equiv 0$ on $[a, b]$ by the uniqueness theorem for ordinary differential equation. We now use a change of variables to transform the Riccati differential equation $(\natural)$ to a linear differential equation. The change of variable is $w = -\frac{u'}{u}P$ so that $(\natural)$ becomes

$$P\left(Q - \frac{(u'P)'}{u} + \frac{u'^2}{u^2}\,P\right) = \frac{u'^2}{u^2}\,P^2,$$

which can be rewritten as

$$(\flat) \qquad -\frac{d}{dx}\left(Pu'\right) + Qu = 0.$$

For this change of variable there is an implicit assumption that u is never zero. Here is the key idea. Assume that no point in $(a, b]$ is conjugate to a. Then when we solve the second-order differential equation $(\flat)$ by using the initial condition $u(a) = 0$ and $u'(a) \neq 0$ (for example $u'(a) = 1$), we can conclude that $u(x)$ is nowhere zero on $(a, b]$. By continuity argument, when we solve the the second-order differential equation $(\flat)$ on $[a-\varepsilon, b]$ with $u(a-\varepsilon) = 0$ and $u'(a-\varepsilon) = 1$ with $\varepsilon > 0$ sufficiently small, $u(x)$ will be nowhere zero on $(a-\varepsilon, b]$ and in particular nowhere zero on $[a, b]$. This means that we are able to find w and conclude that

$$\int_a^b \left(P(x)h'(x)^2 + Q(x)h(x)^2\right) dx > 0$$

for all $h(x)$ with $h(a) = h(b) = 0$.

Sufficient Condition for Local Minimum. If $P(x) > 0$ on $[a, b]$ and if there is no point in $(a, b]$ conjugate to a, then the extremal function is a local minimum.