Math 115 Final Examination with Solutions
JANUARY 22, 2007, 2:15 P.M. - 5:15 P.M.
SEVER 107

Some formulae are provided at the end of the problem list. Not all of them are necessary for solving the problems.

Problem 1. Use the theory of residues to evaluate the following three definite integrals. Justify all the steps and the limits of the integrals occurring in the computation. (*Hint:* use as contour of integration the boundary of the upper half disk centered and indented at the origin with its radius going to infinity and the indentation shrinking down to the origin. The three integrals can be computed by the same techniques with only small variations among the three computations.)

(a) (6 *points*)
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx,$$

where the integral means the limit of the integral from $-R$ to R as $R \to \infty$.

(b) (6 *points*)
$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2}\, dx.$$

(c) (6 *points*)
$$\int_{-\infty}^{\infty} \frac{1 - 2\sin x + \sin 2x}{x^3}\, dx,$$

where the integral means the limit of the integral over $[-R, -r] \cup [r, R]$ for $0 < r < R$ as $r \to 0$ and $R \to \infty$.

SOLUTION. **(a)** First of all let us make some observations which will be used in all three parts (a), (b), and (c). Let C_R be the half-circle consisting of all points $Re^{i\theta}$ with $0 \leq \theta \leq \pi$. Let $P(z)$ and $Q(z)$ be two polynomials such that that the degree of $Q(z)$ may only be one more than that of $P(z)$. Our observation is that for any positive integer k,

$$\int_{C_R} \frac{P(z)e^{ikz}}{Q(z)} dz \to 0 \text{ as } R \to 0$$

which is verified as follows by integration by parts with integration applied to the factor e^{ikz}.

$$\int_{C_R} \frac{P(z)e^{ikz}}{Q(z)} dz$$

$$= \left. \frac{P(z)e^{ikz}}{ik\,Q(z)} \right|_{z=-R}^{R} - \int_{C_R} \left(\frac{d}{dz} \frac{P(z)}{Q(z)} \right) e^{ikz} dz$$

$$= \left. \frac{P(z)e^{ikz}}{ik\,Q(z)} \right|_{z=-R}^{R} - \int_{C_R} \frac{\left(P'(z)Q(z) - P(z)Q'(z)\right) e^{ikz}}{Q(z)^2} dz.$$

The we use

$$\left| \frac{(P'(z)Q(z) - P(z)Q'(z))}{Q(z)^2} \right| = O\left(\frac{1}{R^2} \right)$$

(from the degree of $P'(z)Q(z) - P(z)Q'(z)$ no more than the degree of $Q(z)^2$ minus 2) and also use

$$\left| e^{ikz} \right| = e^{-k(\operatorname{Im} z)} \leq 1$$

(from $\operatorname{Im} z > 0$ on C_R).

The second observation is that for a meromorphic function $f(z)$ with a simple pole at the origin,

$$\lim_{r \to 0} \int_{C_r} f(z) dz = \pi\, i\, \operatorname{Res}_{z=0} f(z).$$

With these two observations, by applying Cauchy's theorem to the integration of holomorphic function $\frac{e^{iz}}{z}\,dz$ over the contour $C_R \cup C_r \cup [-R, -r] \cup [r, R]$ in the counterclockwise sense and letting $r \to 0$ and $R \to \infty$, we obtain

$$\int_{x=-\infty}^{\infty} \frac{e^{ix}}{x} dx = \lim_{r \to 0} \int_{C_r} \frac{e^{iz}}{z} dz = \pi\, i\, \operatorname{Res}_{z=0} \frac{e^{iz}}{z} = \pi i,$$

where the integral on the right-hand side is interpreted as the limit of the integral over $[-R, -r] \cup [r, R]$ with $0 < r < R$ as $r \to 0$ and $R \to \infty$. Since

$$\int_{x=-\infty}^{\infty} \frac{\sin x}{x} dx = \operatorname{Im} \int_{x=-\infty}^{\infty} \frac{e^{ix}}{x} dx,$$

it follows that

$$\int_{x=-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

(b) The only modification from Part (a) is that $f(z)$ is now $\frac{1-e^{iz}}{z^2}$ and at the end we have to take the real part instead of the imaginary part. The answer is the real part of the product of πi and the residue of $\frac{1-e^{iz}}{z^2}$ at the origin which is equal to

$$\operatorname{Re}\left(\pi i \operatorname{Res}_{z=0}\frac{1-e^{iz}}{z^2}\right)=\operatorname{Re}\left(\pi i \operatorname{Res}_{z=0}\frac{1-(1+iz+\cdots)}{z^2}\right)=\pi.$$

(c) The only modification from Part (a) is that $f(z)$ is now $\frac{1-2e^{iz}+e^{i2z}}{z^3}$. The answer is the imaginary part of the product of πi and the residue of $\frac{1-2e^{iz}+e^{i2z}}{z^3}$ at the origin which is equal to

$$\begin{aligned}&\operatorname{Im}\left(\pi i \operatorname{Res}_{z=0}\frac{1-2e^{iz}+e^{i2z}}{z^3}\right)\\ =&\operatorname{Im}\left(\pi i \operatorname{Res}_{z=0}\frac{1-2\left(1+iz-\frac{z^2}{2}+\cdots\right)+(1+i2z-2z^2+\cdots)}{z^3}\right)\\ =&-\pi.\end{aligned}$$

Note that

$$\int_{-r}^{r}\frac{dx}{x^3}=\left[-\frac{1}{2x^2}\right]_{x=-r}^{r}=0,$$

which in particular implies that

$$\lim_{r\to 0}\int_{-r}^{r}\frac{dx}{x^3}=0.$$

The integral

$$\int_{-\infty}^{\infty}\frac{1-2\sin x+\sin 2x}{x^3}\,dx$$

in the problem, which is interpreted as the limit of the integral over $[-R,-r]\cup[r,R]$ for $0<r<R$ as $r\to 0$ and $R\to\infty$, can actually be formulated more easily as

$$\int_{-\infty}^{\infty}\frac{-2\sin x+\sin 2x}{x^3}\,dx$$

which is absolutely integrable over $(-\infty,\infty)$ and whose integrand is continuous at $x=0$. The reason why 1 is added to the numerator is meant to be as a hint that the meromorphic function $\frac{1-2e^{iz}+e^{i2z}}{z^3}$ with a simple pole at the origin is to be used.

Problem 2. Use the theory of residues to evaluate the following three definite integrals. (*Hint:* use as contour of integration the boundary of the disk centered at the origin with a small disk puncture at the origin and a slit along the positive real axis as the radius of the disk goes to infinity and the puncture shrinks down to the origin. The three integrals can be computed by the same techniques with only small variations among the three computations.)

(a) (6 *points*)

$$\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx \quad \text{for} \quad 0 < \alpha < 1.$$

(b) (6 *points*)

$$\int_0^\infty \frac{x^{-\alpha}\log x}{x+1}\,dx \quad \text{for} \quad 0 < \alpha < 1.$$

Hint: Use the result from Part (a).

(c) (6 *points*)

$$\int_0^\infty \frac{x^{-\alpha}\left(\log x\right)^2}{x+1}\,dx \quad \text{for} \quad 0 < \alpha < 1.$$

Hint: Use the results from Part (a) and Part (b).

SOLUTION. **(a)** For the evaluation of

$$\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx$$

we use a branch of the function $z^{-\alpha}$ defined by polar coordinates as follows. For $z = re^{i\theta}$ with $0 \le \theta \le 2\pi$ the value of $z^{-\alpha}$ is $r^{-\alpha}e^{-i\alpha\theta}$.

Choose $0 < r < R$. Let C_R be the curve $\theta \mapsto Re^{i\theta}$ for $0 \le \theta \le 2\pi$ and C_r be the curve $\theta \mapsto re^{i\theta}$ for $0 \le \theta \le 2\pi$.

We now integrate

$$f(z) := \frac{z^{-\alpha}}{z+1}$$

over the contour which starts from the real point r, goes along the real axis from r to R and then follows the curve C_R in the counterclockwise sense to get from the real point R back to the same real point R and then goes along the real axis from R to r and then follows the curve C_r in the clockwise sense to get from the real point r back to the same real point r.

We calculate the residue $\mathrm{Res}_{z=-1} f(z)$, which is given by

$$\lim_{z\to -1}(z-(-1))\,f(z)=\lim_{z\to -1}(z-(-1))\,\frac{z^{-\alpha}}{z+1}=\lim_{z\to -1}z^{-\alpha}=e^{-i\alpha\pi}.$$

By the residue theorem over the contour described above,

$$\int_r^R \frac{x^{-\alpha}}{x+1}\,dx+\int_{C_R} f(z)dz-\int_{C_r} f(z)dz-\int_r^R \frac{e^{-i\alpha 2\pi}x^{-\alpha}}{x+1}\,dx$$
$$=2\pi i\,\mathrm{Res}_{z=-1}f(z)=2\pi i\,e^{-i\alpha\pi}.$$

The reason for the last integral on the left-hand side of the equation is that the value of $z^{-\alpha}$ is $x^{-\alpha}e^{-i\alpha 2\pi}$ at the real point x, because at the real point x the angle θ for its polar representation is 2π. As $R\to\infty$,

$$\left|\int_{C_R} f(z)dz\right|\le \sup_{z\in C_R}|f(z)|\cdot(\text{length of } C_R)\le \frac{R^{-\alpha}}{R-1}\cdot 2\pi R$$

approaches 0, because $\alpha>0$. As $r\to 0$,

$$\left|\int_{C_r} f(z)dz\right|\le \sup_{z\in C_r}|f(z)|\cdot(\text{length of } C_r)\le \frac{r^{-\alpha}}{1-r}\cdot 2\pi r$$

approaches 0, because $\alpha<1$. Hence

$$\left(1-e^{-i\alpha 2\pi}\right)\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx=2\pi i\,e^{-i\alpha\pi}$$

and

$$\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx=\frac{2\pi i\,e^{-i\alpha\pi}}{1-e^{-i\alpha 2\pi}}=\frac{2\pi i}{e^{i\alpha\pi}-e^{-i\alpha\pi}}=\frac{\pi}{\sin\pi\alpha}.$$

(b) The same argument gives us

$$\int_0^\infty \frac{x^{-\alpha}\log x}{x+1}\,dx-\int_0^\infty \frac{e^{-i\alpha 2\pi}x^{-\alpha}\left(\log x+2\pi i\right)}{x+1}\,dx$$
$$=2\pi i\,\mathrm{Res}_{z=-1}\frac{z^{-\alpha}\log z}{z+1}=2\pi i\,e^{-i\alpha\pi}\pi i$$

and

$$\left(1-e^{-i\alpha 2\pi}\right)\int_0^\infty \frac{x^{-\alpha}\log x}{x+1}\,dx-\int_0^\infty \frac{e^{-i\alpha 2\pi}x^{-\alpha}\,2\pi i}{x+1}\,dx=2\pi i\,e^{-i\alpha\pi}\pi i.$$

Thus

$$\int_0^\infty \frac{x^{-\alpha}\log x}{x+1}\,dx = \frac{2\pi i\, e^{-i\alpha\pi}\pi i}{1-e^{-i\alpha 2\pi}} + \frac{e^{-i\alpha 2\pi}2\pi i}{1-e^{-i\alpha 2\pi}}\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx$$
$$= \frac{\pi}{\sin\pi\alpha}\pi i + \frac{\pi e^{-i\alpha\pi}}{\sin\pi\alpha}\frac{\pi}{\sin\pi\alpha} = \frac{\pi^2\cos\pi\alpha}{\sin^2\pi\alpha}.$$

(c) The same argument gives us

$$\int_0^\infty \frac{x^{-\alpha}(\log x)^2}{x+1}\,dx - \int_0^\infty \frac{e^{-i\alpha 2\pi}x^{-\alpha}(\log x + 2\pi i)^2}{x+1}\,dx$$
$$= 2\pi i\,\mathrm{Res}_{z=-1}\frac{z^{-\alpha}(\log z)^2}{z+1} = -2\pi i\, e^{-i\alpha\pi}\pi^2$$

and

$$\left(1-e^{-i\alpha 2\pi}\right)\int_0^\infty \frac{x^{-\alpha}(\log x)^2}{x+1}\,dx - 4\pi i\int_0^\infty \frac{e^{-i\alpha 2\pi}x^{-\alpha}\log x}{x+1}\,dx$$
$$+4\pi^2\int_0^\infty \frac{e^{-i\alpha 2\pi}x^{-\alpha}}{x+1}\,dx = -2\pi i\, e^{-i\alpha\pi}\pi^2.$$

Thus

$$\int_0^\infty \frac{x^{-\alpha}(\log x)^2}{x+1}\,dx$$
$$= \frac{2\pi i\, e^{-i\alpha\pi}\pi i}{1-e^{-i\alpha 2\pi}} + \frac{e^{-i\alpha 2\pi}4\pi i}{1-e^{-i\alpha 2\pi}}\int_0^\infty \frac{x^{-\alpha}\log x}{x+1}\,dx - \frac{e^{-i\alpha 2\pi}4\pi^2}{1-e^{-i\alpha 2\pi}}\int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx$$
$$= -\frac{\pi}{\sin\pi\alpha}\pi^2 + \frac{2\pi e^{-i\alpha\pi}}{\sin\pi\alpha}\frac{\pi^2\cos\pi\alpha}{\sin^2\pi\alpha} + \frac{2\pi^2 i e^{-i\alpha\pi}}{\sin\pi\alpha}\frac{\pi}{\sin\pi\alpha}$$
$$= \frac{\pi^3\left(-\sin^2\pi\alpha + 2\cos\pi\alpha\,(\cos\pi\alpha - i\sin\pi\alpha) + i2\sin\pi\alpha\,(\cos\pi\alpha - i\sin\pi\alpha)\right)}{\sin^3\pi\alpha}$$
$$= \frac{\pi^3\left(1+\cos^2\pi\alpha\right)}{\sin^3\pi\alpha}.$$

Problem 3. Let D be the domain in the plane $\mathbb{C}$ of complex numbers which is the intersection between the upper half-plane and the open disk of radius 1 centered at the point $(x,y) = \left(0,\frac{1}{2}\right)$. In other words,

$$D = \left\{ z = x+iy \in \mathbb{C} \,\middle|\, y>0,\ \left|z-\frac{i}{2}\right| < 1 \right\}.$$

(a) *(7 points)* Use conformal maps to write down explicitly the harmonic function $u(z)$ on D which assumes the boundary value 1 on the circular part of the boundary and the boundary value 0 on the part of the boundary lying on the x-axis. Calculate the value $u\left(\frac{i}{2}\right)$ of the harmonic function $u(z)$ at the point $(x, y) = \left(0, \frac{1}{2}\right)$.

(b) *(7 points)* Consider a 2-dimensional steady irrotational incompressible fluid flow of constant density in D. The fluid enters through a slit represented by the point $(x, y) = \left(-\frac{\sqrt{3}}{2}, 0\right)$ of intersection of the circle $\left|z - \frac{i}{2}\right| = 1$ and the real axis at the rate of Q units per unit time so that the flow exits at the other point $(x, y) = \left(\frac{\sqrt{3}}{2}, 0\right)$ of intersection of the circle $\left|z - \frac{i}{2}\right| = 1$ and the real axis at the rate of Q units per unit time. Find the flow lines and compute the velocity of the flow at the point $(x, y) = \left(0, \frac{1}{2}\right)$.

Solution. **(a)** Apply the linear fractional transformation

$$w = \frac{z - \frac{\sqrt{3}}{2}}{z + \frac{\sqrt{3}}{2}}$$

to conformally map the domain D in the z-plane to the domain Ω in w-plane which is the sector with vertex at the origin and bounded by the two rays $\arg w = \frac{\pi}{3}$ and $\arg w = \pi$. The function u is now a linear combination of 1 and $\arg w$ and is given by

$$u = \frac{\pi - \arg w}{\pi - \frac{\pi}{3}} = \frac{3}{2}\left(1 - \frac{1}{\pi}\arg w\right)$$

with the value of $\arg w$ defined by $0 < \arg w < 2\pi$. The final answer is

$$u = \frac{3}{2}\left(1 - \frac{1}{\pi}\arg\left(\frac{2z - \sqrt{3}}{2z + \sqrt{3}}\right)\right)$$

with the value of arg inside $(0, 2\pi)$. The value $u\left(\frac{i}{2}\right)$ of $u(z)$ at $z = \frac{i}{2}$ is given by

$$u\left(\frac{i}{2}\right) = \frac{3}{2}\left(1 - \frac{1}{\pi}\arg\left(\frac{i - \sqrt{3}}{i + \sqrt{3}}\right)\right)$$
$$= \frac{3}{2}\left(1 - \frac{1}{\pi}\frac{2\pi}{3}\right) = \frac{3}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{2}.$$

(b) We use the same notations as in Part (a). The streamlines are given by $\arg w$ being constant. Thus the complex potential is given by a real constant times $\log w$. At the sink at the point $z = \frac{\sqrt{3}}{2}$ the two banks are given by $\arg w = \frac{\pi}{3}$ and $\arg w = \pi$. If the direction of the flow is reversed, the streamline $\arg w = \frac{\pi}{3}$ is on the right of the flow and the streamline line $\arg w = \pi$ is on the left of the flow and the difference of two constants is $\frac{2\pi}{3}$. Since the magnitude of the sink is Q, if the flow is reversed, the complex potential function should be

$$\frac{Q}{\frac{2\pi}{3}} \log w.$$

Without the reversal of the direction of the flow, the complex potential function for the original flow should be

$$F(w) = -\frac{Q}{\frac{2\pi}{3}} \log w = -\frac{Q}{\frac{2\pi}{3}} \log\left(\frac{z - \frac{\sqrt{3}}{2}}{z + \frac{\sqrt{3}}{2}}\right) = -\frac{3Q}{2\pi} \log\left(\frac{2z - \sqrt{3}}{2z + \sqrt{3}}\right).$$

The streamlines are given by

$$\arg\left(\frac{2z - \sqrt{3}}{2z + \sqrt{3}}\right) = \text{constant}$$

and the velocity at the point z is given by

$$\overline{\frac{dF}{dz}(z)} = -\frac{3Q}{2\pi} \frac{2\bar{z} + \sqrt{3}}{2\bar{z} - \sqrt{3}} \frac{4\sqrt{3}}{\left(2\bar{z} + \sqrt{3}\right)^2} = -\frac{6\sqrt{3}\,Q}{\pi\left(4\bar{z}^2 - 3\right)}.$$

At the point $z = \frac{i}{2}$ the velocity of the flow as a complex number is given by

$$-\left.\frac{6\sqrt{3}\,Q}{\pi\left(4\bar{z}^2 - 3\right)}\right|_{z=\frac{i}{2}} = -\frac{6\sqrt{3}\,Q}{\pi\left(4\overline{\left(\frac{i}{2}\right)}^2 - 3\right)} = \frac{3\sqrt{3}\,Q}{\pi}.$$

Problem 4. Evaluate the following definite integral

$$\int_0^{2\pi} \frac{d\theta}{a - \cos\theta - \sin\theta} \quad \text{for} \quad a > 2$$

in the two ways described below. For this problem you are required to work out in detail the two numerical answers and check that they agree and not just write down the formulas. (*Hint:* use the addition formula for $\cos(A+B)$.)

(a) *(6 points)* Use the parametrization of the unit circle

$$z = e^{i\theta} \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$

to transform the integral to a contour integral over the unit circle and then use the theory of residues to compute the definite integral.

(b) *(6 points)* Transform the integrand of the definite integral to the value of the Poisson kernel at one point and then use the Poisson integral formula to evaluate the definite integral.

(c) *(6 points)* Use the method in Part (b) and a harmonic measure to evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{a - \cos\theta - \sin\theta} \quad \text{for} \quad a > 2.$$

SOLUTION. **(a)** We apply the addition formula to get

$$\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$$

and

$$\frac{1}{a - \cos\theta - \sin\theta} = \frac{1}{a - \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)}.$$

We now use the parametrization $z = e^{i\theta}$ so that

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z + \frac{1}{z}}{2}$$

and the integral

$$\int_0^{2\pi} \frac{d\theta}{a - \cos\theta - \sin\theta} = \int_0^{2\pi} \frac{d\theta}{a - \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)}$$

is transformed to the contour integral

$$\oint_{|z|=1} \frac{\frac{dz}{iz}}{a - \sqrt{2}\,\frac{z+\frac{1}{z}}{2}} = i\oint_{|z|=1} \frac{\sqrt{2}\,dz}{z^2 - \sqrt{2}az + 1}.$$

We locate the pole of the denominator of the integrand within the unit disk by solving the quadratic equation

$$z^2 - \sqrt{2}az + 1 = 0$$

and get
$$z = \frac{\sqrt{2}a \pm \sqrt{2a^2 - 4}}{2}.$$
Since from $a > \sqrt{2}$ we know that the point
$$\frac{\sqrt{2}a + \sqrt{2a^2 - 4}}{2}$$
has absolute value $\frac{\sqrt{2}a}{2} > 1$ and sine we know that the integral
$$\int_0^{2\pi} \frac{d\theta}{a - \cos\theta - \sin\theta} = \int_0^{2\pi} \frac{d\theta}{a - \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)}$$
we are going to evaluate has positive integrand and cannot be zero, the other zero
$$z = \frac{\sqrt{2}a - \sqrt{2a^2 - 4}}{2}$$
must have absolute value < 1. Let
$$w = \frac{\sqrt{2}a - \sqrt{2a^2 - 4}}{2} \quad \text{and} \quad w' = \frac{\sqrt{2}a + \sqrt{2a^2 - 4}}{2}.$$
Then
$$i \oint_{|z|=1} \frac{\sqrt{2}\, dz}{z^2 - \sqrt{2}az + 1} = i \oint_{|z|=1} \frac{\sqrt{2}\, dz}{(z - w)(z - w')}$$
which by the theory of residues equals
$$2\pi i\, i \frac{\sqrt{2}}{w - w'} = 2\pi i\, i \frac{\sqrt{2}}{-\sqrt{2a^2 - 4}} = \frac{2\pi}{\sqrt{a^2 - 2}}.$$
(b) As in Part (a) we apply the addition formula to get
$$\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$$
and
$$\frac{1}{a - \cos\theta - \sin\theta} = \frac{1}{a - \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)}.$$
We find $0 \le r < 1$ so that
$$1 + r^2 - 2r\cos\left(\theta - \frac{\pi}{4}\right).$$

is equal to a constant times

$$a - \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right).$$

It means that the constant must be $\sqrt{2}r$ and we must have

$$\sqrt{2}r\,a = 1 + r^2.$$

We now determine $0 \leq r < 1$ by solving the equation

$$r^2 - \sqrt{2}r\,a + 1 = 0$$

to get

$$r = \frac{a\sqrt{2} - \sqrt{2a^2 - 4}}{2}.$$

We now check that $r < 1$ by observing that

$$r = \frac{a\sqrt{2} - \sqrt{2a^2 - 4}}{2} < 1$$

is equivalent to

$$a\sqrt{2} < 2 + \sqrt{2a^2 - 4}$$

and the squaring of both sides make it equivalent to

$$2a^2 < 4 + 4\sqrt{2a^2 - 4} + \left(2a^2 - 4\right) = 2a^2 + 4\sqrt{2a^2 - 4}$$

which clearly holds when $a > \sqrt{2}$. Another way to see

$$r = \frac{a\sqrt{2} - \sqrt{2a^2 - 4}}{2} < 1$$

is to multiply both the numerator and the denominator by $a\sqrt{2} + \sqrt{2a^2 - 4}$ to get

$$r = \frac{2a^2 - (2a^2 - 4)}{2\left(a\sqrt{2} + \sqrt{2a^2 - 4}\right)} = \frac{2}{a\sqrt{2} + \sqrt{2a^2 - 4}}$$

which is < 1 when $a > \sqrt{2}$. To the integral

$$\int_0^{2\pi} \frac{d\theta}{a - \cos\theta - \sin\theta} = \sqrt{2}r \int_0^{2\pi} \frac{d\theta}{1 + r^2 - 2r\cos\left(\theta - \frac{\pi}{4}\right)}$$

we apply Poisson integral formula for the function which is identically 1 to get

$$1 = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\left(\theta - \frac{\pi}{4}\right)}\, d\theta.$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{a - \cos\theta - \sin\theta} = \frac{2\pi\sqrt{2}r}{1 - r^2}$$

with

$$r = \frac{a\sqrt{2} - \sqrt{2a^2 - 4}}{2}.$$

To reconcile this with the expression in Part (a), we let $b = \sqrt{a^2 - 2}$ so that

$$r = \frac{a - b}{\sqrt{2}}$$

and $a^2 + b^2 + 2$. Then

$$\frac{r}{1 - r^2} = \frac{\frac{a-b}{\sqrt{2}}}{1 - \frac{a^2 - 2ab + b^2}{2}}$$

$$= \frac{\sqrt{2}(a - b)}{2 - a^2 + 2ab - b^2} = \frac{\sqrt{2}(a - b)}{2ab - b^2} = \frac{1}{\sqrt{2}b}$$

and we get the expression

$$\int_0^{2\pi} \frac{d\theta}{a - \cos\theta - \sin\theta} = \frac{2\pi}{b} = \frac{2\pi}{\sqrt{a^2 - 4}}.$$

(c) We introduce the harmonic measure $\omega_{\alpha,\beta}$ from the formulae given at the end of the problem list which is characterized by the following properties:

(i) $\omega_{\alpha,\beta}$ is harmonic on the open unit disk,

(ii) $\omega_{\alpha,\beta}$ assumes the value 1 on the arc in the counterclockwise sense between $e^{i\alpha}$ and $e^{i\beta}$, and

(iii) $\omega_{\alpha,\beta}$ assumes the value 0 on the arc in the counterclockwise sense between $e^{i\beta}$ and $e^{i\alpha}$.

The formula for $\omega_{\alpha,\beta}$ is

$$\omega_{\alpha,\beta}(z)=\frac{1}{\pi}\left(\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right)-\frac{1}{2}\left(\beta-\alpha\right)\right),$$

where the argument function is defined with cuts

$$\left\{\rho e^{i\alpha}\,\middle|\,\rho\geq 1\right\},\quad\left\{\rho e^{i\beta}\,\middle|\,\rho\geq 1\right\}.$$

We use the value

$$r=\frac{a\sqrt{2}-\sqrt{2a^2-4}}{2}$$

from Part (b) and set $\alpha=\frac{\pi}{4}$ and $\beta=\frac{\pi}{2}$. Then

$$\omega_{\frac{\pi}{4},\frac{\pi}{2}}\left(re^{i\frac{\pi}{4}}\right)=\frac{1}{2\pi}\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1-r^2}{1+r^2-2r\cos\left(\theta-\frac{\pi}{4}\right)}\,d\theta.$$

Since from the choice of r which gives $\sqrt{2}r\,a=1+r^2$ we have

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\theta}{a-\cos\theta-\sin\theta}=\sqrt{2}r\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\theta}{1+r^2-2r\cos\left(\theta-\frac{\pi}{4}\right)},$$

it follows that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\theta}{a-\cos\theta-\sin\theta}=\frac{2\pi\sqrt{2}r}{1-r^2}\,\omega_{\frac{\pi}{4},\frac{\pi}{2}}\left(re^{i\frac{\pi}{4}}\right).$$

We now compute

$$\arg\left(\frac{re^{i\frac{\pi}{4}}-e^{i\frac{\pi}{2}}}{re^{i\frac{\pi}{4}}-e^{i\frac{\pi}{4}}}\right),$$

whose value is equal to the angle at $re^{i\frac{\pi}{4}}$ of the triangle with vertices $e^{i\frac{\pi}{4}},re^{i\frac{\pi}{4}},e^{i\frac{\pi}{2}}$ and is equal to

$$\left(\pi-\tan^{-1}\left(\frac{1-\frac{r}{\sqrt{2}}}{\frac{r}{\sqrt{2}}}\right)\right)-\frac{\pi}{4}.$$

Finally from the formula for harmonic measure

$$\omega_{\frac{\pi}{4},\frac{\pi}{2}}\left(re^{i\frac{\pi}{4}}\right)=\frac{1}{\pi}\left(\arg\left(\frac{re^{i\frac{\pi}{4}}-e^{i\frac{\pi}{2}}}{re^{i\frac{\pi}{4}}-e^{i\frac{\pi}{4}}}\right)-\frac{\pi}{8}\right),$$

we get the final answer

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{a-\cos\theta-\sin\theta} = \frac{2\pi\sqrt{2}r}{1-r^2}\,\omega_{\frac{\pi}{4},\frac{\pi}{2}}\left(re^{i\frac{\pi}{4}}\right)$$

$$= \frac{2\pi\sqrt{2}r}{1-r^2}\left(\frac{5}{8}-\frac{1}{\pi}\tan^{-1}\left(\frac{\sqrt{2}-r}{r}\right)\right),$$

where

$$r = \frac{a\sqrt{2}-\sqrt{2a^2-4}}{2}.$$

Problem 5. Evaluate the following infinite series

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

in the two ways described below. For this problem you are required to work out in detail the two numerical answers and check that they agree and not just write down the formulas.

(a) *(6 points)* Use the Fourier series of the even function $f(x) = x^2$ for $-\pi \leq x \leq \pi$ and the Parseval identity.

(b) *(7 points)* Use the theory of residues and the cotangent function to evaluate

$$\int_{C_n} \frac{\pi\,\cot\pi z}{z^4}dz,$$

where C_n is the square with corners at

$$\left(n+\frac{1}{2}\right)(\pm 1 \pm i).$$

Hint: The residue of $\frac{\pi\,\cot\pi z}{z^4}$ at a nonzero integer n is $\frac{1}{n^4}$. Compute the residue of $\frac{\pi\,\cot\pi z}{z^4}$ at the pole $z=0$ which is of order 5 and is *not* simple. Use the following power series expansion of

$$z\cot z = 1 + \frac{B_2}{2!}(2z)^2 + \frac{B_4}{4!}(2z)^4 + \cdots,$$

where $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$.

SOLUTION. **(a)** Since the function $f(x) = x^2$ is even, the only possible nonzero Fourier coefficients are a_n.

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_{x=0}^{x=\pi} = \frac{2\pi^2}{3}$$

and for any positive integer n we use integration by parts to get

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x^2 \cos nx\, dx \\ &= -\frac{4}{\pi n}\int_0^\pi x \sin nx\, dx \\ &= \frac{4}{\pi n^2}\Big[x\cos nx\Big]_{x=0}^{x=\pi} - \frac{4}{\pi n^2}\int_0^\pi \cos nx\, dx \\ &= \frac{4}{n^2}\cos n\pi = (-1)^n \frac{4}{n^2}. \end{aligned}$$

Thus we have the Fourier series expansion

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^\infty (-1)^n \frac{4}{n^2}\cos nx.$$

We now use the following identity of Parseval

$$\frac{1}{\pi}\int_{-\pi}^\pi (f(x))^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^\infty \left(a_n^2 + b_n^2\right)$$

to get

$$\frac{1}{\pi}\int_{-\pi}^\pi x^4\, dx = \frac{2\pi^4}{9} + \sum_{n=1}^\infty \frac{16}{n^4}.$$

Thus

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{n^4} &= \frac{1}{16}\left(\frac{1}{\pi}\int_{-\pi}^\pi x^4\, dx - \frac{2\pi^4}{9}\right) \\ &= \frac{1}{16}\left(\frac{2\pi^5}{5} - \frac{2\pi^4}{9}\right) = \frac{\pi^4}{90} \end{aligned}$$

(b) The residue of $\frac{\pi \cot \pi z}{z^4}$ at $z = 0$ is the coefficient of $\frac{1}{z}$ in

$$\frac{\pi \cot \pi z}{z^4} = \frac{1}{z^5}\left(1 + \frac{B_2}{2!}(2\pi z)^2 + \frac{B_4}{4!}(2\pi z)^4 + \cdots\right)$$

which is equal to

$$\frac{B_4}{4!}\left(2\pi\right)^4 = -\frac{\pi^4}{45}.$$

Since

$$\begin{aligned} 0 &= \lim_{n\to\infty}\int_{C_n}\frac{\pi\,\cot\pi z}{z^4}dz \\ &= 2\pi i\left(\sum_{n=-\infty}^{\infty}\frac{1}{n^4}-\frac{\pi^4}{45}\right), \end{aligned}$$

it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

Problem 6. For $x_2 > x_1 > 0$ consider the functional

$$(\dagger)\qquad J[y] = \int_{x_1}^{x_2} F\left(x, y, y'\right)$$

with

$$F\left(x, y, y'\right) = x^2\, y'^2 + 12y^2.$$

Let $y_1 = y\left(x_1\right)$ and $y_2 = y\left(x_2\right)$.

(a) *(11 points)* Write down the Euler-Lagrange equation for the variation of the functional $J[y]$ and solve it to get a function $y = y(x)$ of the variable x with two yet-to-be-determined constants a and b in it.

Hint: Consider special solutions $y = x^n$ for some constant n.

(b) *(11 points)* For the case where $x_1 = 1$, $x_2 = 2$, $y_1 = 0$, and $y_2 = 1$, compute the extremal function $y = y(x)$ for the functional $J[y]$ with the two given end-points (x_1, y_1) and (x_2, y_2).

(c) *(11 points)* For the case where $x_1 = 1$, $x_2 = 2$, and $y_1 = 0$, and y_2 is allowed to vary freely, compute the extremal function $y = y(x)$ for the functional $J[y]$.

(d) *(12 points)* For the case where $x_1 = 1$, $x_2 = 2$, $y_1 = 0$, and $y_2 = 1$, compute the extremal function $y = y(x)$ for the functional $J[y]$ with the two given end-points (x_1, y_1) and (x_2, y_2), subject to the integral constraint

$$\int_{x_1}^{x_2} 2y\, dx = 1.$$

(e) *(12 points)* Show that the extremal $y = y(x)$ for $J[y]$ obtained in Part (b) with fixed end-points $(1, 0)$ and $(1, 2)$ is a local (weak) minimum by verifying that for the Jacobi equation of the functional $J[y]$ there is no point (x, y) on the extremal with $1 < x \leq 2$ conjugate to the point $(1, 0)$ and that the strengthened Legendre condition is satisfied.

(f) *(12 points)* Write down the Hamiltonian

$$H\left(x, y, p\right) = y' p - F\left(x, y, y'\right)$$

for the functional $J[y]$ of (†), where $p = F_{y'}$. Using the action integral to find explicitly a solution $S\left(x, y, \alpha\right)$ of the Hamilton-Jacobi differential equation

$$\frac{\partial S}{\partial x} + H\left(x, y, \frac{\partial S}{\partial y}\right) = 0,$$

where α is a constant parameter, such that

$$\frac{\partial^2 S}{\partial y \partial \alpha}$$

is not identically zero.

Solution. **(a)** From

$$F\left(x, y, y'\right) = x^2\, {y'}^2 + 12 y^2$$

we get

$$F_y = 24y, \quad F_{y'} = 2x^2 y'.$$

Hence the Euler-Lagrange equation is

$$(\natural) \qquad 24y - \left(2x^2 y'\right)' = 0$$

or

$$x^2\, y'' + 2xy' - 12y = 0.$$

Consider the special solution $y = x^n$ with n being a constant and the differential equation is reduced to

$$x^2 n(n-1)x^{n-2} + 2xnx^{n-1} - 12x^n = 0.$$

Since we are only interested in the range $x \geq 1$ for x, we can divide out by x^n and get $n(n-1) + 2n - 12 = 0$ or $n^2 + n - 12 = 0$ which factors into $(n+4)(n-3) = 0$ and gives us the two solutions $n = 3$ and $n = -4$. The general solution of the Euler-Lagrange equation with two yet-to-be-determined constants a and b is

$$y = ax^3 + \frac{b}{x^4}.$$

(b) From the boundary condition $x_1 = 1$ and $y_1 = 0$ we get $b = -a$ and the solution becomes

$$y = a\left(x^3 - \frac{1}{x^4}\right)$$

with only one yet-to-be-determined constant a. The other condition $x_2 = 2$ and $y_2 = 1$ gives

$$a\left(8 - \frac{1}{16}\right) = 1$$

and $a = \frac{16}{127}$. The extremal is

$$y = \frac{16}{127}\left(x^3 - \frac{1}{x^4}\right).$$

(c) When y_2 is free to vary, the condition is $F_{y'} = 0$, which means $F_{y'} = 2x^2y' = 0$. For our case of

$$y = a\left(x^3 - \frac{1}{x^4}\right)$$

and

$$y' = a\left(3x^2 + \frac{4}{x^5}\right)$$

we get

$$2x^2a\left(3x^2 + \frac{4}{x^5}\right) = 0$$

for $x = 2$, which gives us $a = 0$. The extremal is $y \equiv 0$.

(d) The Euler-Lagrange equation for the situation of integral constraint

$$\int_{x_1}^{x_2} 2y\,dx = 1$$

is

$$12y - \left(2x^2y'\right)' = \lambda,$$

where λ be the Lagrange multiplier. The differential equation is

$$x^2 y'' + 2xy' - 12y + \lambda = 0.$$

Clearly $y \equiv \frac{\lambda}{12}$ is a particular solution of this inhomogeneous differential equation. The general solution is

$$y = ax^3 + \frac{b}{x^4} + \frac{\lambda}{12}$$

with the three constants a, b, λ yet to be determined. The three conditions which determine the three constants are the two boundary conditions $x_1 = 1$, $x_2 = 2$, $y_1 = 0$, $y_2 = 1$ and the integral constraint

$$\int_{x_1}^{x_2} 2y\,dx = 1.$$

We compute the integral

$$\begin{aligned}
\int_1^2 2y\,dx &= \int_1^2 2\left(ax^3 + \frac{b}{x^4} + \lambda\frac{\lambda}{12}\right) dx \\
&= 2\left[\frac{ax^4}{4} - \frac{b}{3x^3} + \frac{\lambda}{12}x\right]_{x=1}^{x=2} \\
&= 2\left(\frac{15a}{4} - b\left(\frac{1}{24} - \frac{1}{3}\right) + \frac{\lambda}{12}\right) \\
&= 2\left(\frac{15a}{4} + \frac{7b}{24} + \frac{\lambda}{12}\right).
\end{aligned}$$

We have the three linear equations

$$\begin{aligned}
a + b + \frac{\lambda}{12} &= 0, \\
8a + \frac{b}{16} + \frac{\lambda}{12} &= 1, \\
\frac{15a}{4} + \frac{7b}{24} + \frac{\lambda}{12} &= \frac{1}{2}
\end{aligned}$$

to solve for a, b, λ. By using $\frac{\lambda}{12} = -a - b$ to eliminate λ, we get the following two linear equations

$$7a - \frac{15b}{16} = 1,$$

$$\frac{11a}{4} - \frac{17b}{24} = \frac{1}{2}$$

or after clearing the denominators

$$\begin{aligned} 112a - 15b &= 16, \\ 66a - 17b &= 12. \end{aligned}$$

Thus we get

$$\begin{aligned} a &= \frac{16 \times 17 - 12 \times 15}{17 \times 112 - 15 \times 66} = \frac{46}{457}, \\ b &= \frac{16 \times 33 - 12 \times 56}{17 \times 56 - 15 \times 33} = -\frac{144}{457}. \end{aligned}$$

We get

$$\frac{\lambda}{12} = -(a+b) = \frac{98}{457}$$

and the extremal is

$$y = \frac{2}{457}\left(23x^3 - \frac{72}{x^4} + 49\right).$$

(e) Since

$$F_{y'y'} = 2x^2 > 0 \quad \text{for} \quad x \geq 1,$$

the Legendre necessary condition for a local minimum is satisfied. To compute the solution h of the Jacobi field equation Jacobi differential equation is

$$-\frac{d}{dx}(Ph') + Qh = 0$$

with

$$P(x) = \frac{1}{2}F_{y'y'}, \qquad Q(x) = \frac{1}{2}\left(F_{yy} - \left(\frac{d}{dx}F_{yy'}\right)\right)$$

and $h(1) = 0$ and $h'(1) \neq 0$, we can differentiate with respect to the parameter a the general solution

$$y = a\left(x^3 - \frac{1}{x^4}\right)$$

passing through $(x, y) = (1, 0)$ and get

$$h(x) = x^3 - \frac{1}{x^4}$$

whose derivative $h'(1)$ at $x = 1$ is $3 + 4 = 7$ and is nonzero. Since

$$h(x) = x^3 - \frac{1}{x^4} > 1 \quad \text{for} \quad x > 1,$$

we conclude that there is no point along the extremal

$$y = \frac{16}{127}\left(x^3 - \frac{1}{x^4}\right) \quad \text{for} \quad 1 \leq x \leq 2$$

of Part (b) which is conjugate to $x = 1$. Thus the extremal of Part (b) is local (weak) minimum.

Remark. An alternative way to get $h(x)$ is to solve directly the differential equation

$$-\frac{d}{dx}\left(Ph'\right) + Qh = 0$$

with

$$P(x) = \frac{1}{2}F_{y'y'} = x^2, \qquad Q(x) = \frac{1}{2}\left(F_{yy} - \left(\frac{d}{dx}F_{yy'}\right)\right) = 12,$$

which means the differential equation

$$-\left(x^2h'\right)' + 12h = 0.$$

This differential equation is the same as the differential equation ($\natural$) earlier in Part (a) (when h is replaced by y) which we already solved in Part (a) to yield

$$h(x) = ax^3 + \frac{b}{x^4},$$

where a and b are constants. To make sure that $h(1) = 0$, we need only choose $b = -a$ to get

$$h(x) = a\left(x^3 - \frac{1}{x^4}\right).$$

To make sure the $h'(1) \neq 0$, we can set $a = 1$ and use

$$h(x) = x^3 - \frac{1}{x^4},$$

which is > 1 for $x > 1$, to conclude that there is no point along the extremal

$$y = \frac{16}{127}\left(x^3 - \frac{1}{x^4}\right) \quad \text{for} \quad 1 \leq x \leq 2$$

of Part (b) which is conjugate to $x = 1$.

(f) It follows from

$$p = F_{y'} = 2x^2 y'$$

that

$$y' = \frac{p}{2x^2}$$

and

$$\begin{aligned} H(x,y,p) &= y'p - F(x,y,y') \\ &= y'p - \left(x^2 {y'}^2 + 12y^2\right) \\ &= \frac{p^2}{2x^2} - \left(\frac{p^2}{4x^2} + 12y^2\right) \\ &= \frac{p^2}{4x^2} - 12y^2. \end{aligned}$$

Introduce the action integral

$$S = \int_{x=x_1}^{x_2} F\left(x, y, y'\right) dx$$

along an extremal $y = y(x)$ with end-points (x_1, y_1) and (x_2, y_2), as a function of x_1, y_1, x_2, y_2. An extremal is necessarily of the form

$$y = ax^3 + \frac{b}{x^4}$$

with $a = a\left(x_1, y_1, x_2, y_2\right)$ and $b = b\left(x_1, y_1, x_2, y_2\right)$ depending on x_1, y_1, x_2, y_2. We set $x_1 = 1$ and $y_1 = \alpha$. We use $y_2 = y(x_2)$ to get

$$(\ddagger) \qquad \alpha = a + b \quad \text{and} \quad y_2 = a x_2^3 + \frac{b}{x_2^4}.$$

From

$$y' = 3ax^2 - \frac{4b}{x^5}$$

it follows that

$$\begin{aligned} F\left(x, y, y'\right) &= x^2 {y'}^2 + 12y^2 \\ &= x^2 \left(3ax^2 - \frac{4b}{x^5}\right)^2 + 12\left(ax^3 + \frac{b}{x^4}\right)^2 \end{aligned}$$

$$= x^2\left(9a^2x^4 - \frac{24ab}{x^3} + \frac{16b^2}{x^{10}}\right) + 12\left(a^2x^6 + \frac{2ab}{x} + \frac{b^2}{x^8}\right)$$
$$= \left(9a^2x^6 - \frac{24ab}{x} + \frac{16b^2}{x^8}\right) + \left(12a^2x^6 + \frac{24ab}{x} + \frac{12b^2}{x^8}\right)$$
$$= 21a^2x^6 + \frac{28b^2}{x^8}$$

and

$$S = \int_{x=1}^{x_2}\left(21a^2x^6 + \frac{28b^2}{x^8}\right)dx$$
$$= 3a^2x^7 - \frac{4b^2}{x^7}\bigg|_{x=1}^{x=x_2}$$
$$= 3a^2\left(x_2^7 - 1\right) + 4b^2\left(1 - \frac{1}{x_2^7}\right).$$

We now use (‡) to solve for a and b in terms of α, x_2, y_2 to get

$$a = \frac{y_2 - \frac{\alpha}{x_2^4}}{x_2^3 - \frac{1}{x_2^4}} = \frac{x_2^7y_2 - \alpha}{x_2^7 - 1},$$
$$b = \frac{\alpha x_2^3 - y_2}{x_2^3 - \frac{1}{x_2^4}} = \frac{\alpha x_2^7 - x_2^4y_2}{x_2^7 - 1}.$$

When we put the values of a and b into S, we get

$$S = 3\left(\frac{x_2^7y_2 - \alpha}{x_2^7 - 1}\right)^2\left(x_2^7 - 1\right) + 4\left(\frac{\alpha x_2^7 - x_2^4y_2}{x_2^7 - 1}\right)^2\left(1 - \frac{1}{x_2^7}\right).$$

By replacing x_2 by x and y_2 by y, we get our final answer

$$S = 3\left(\frac{x^7y - \alpha}{x^7 - 1}\right)^2\left(x^7 - 1\right) + 4\left(\frac{\alpha x^7 - x^4y}{x^7 - 1}\right)^2\left(1 - \frac{1}{x^7}\right)$$
$$= \frac{3\left(x^7y - \alpha\right)^2}{x^7 - 1} + \frac{4\left(\alpha x^7 - x^4y\right)^2}{x^7\left(x^7 - 1\right)}$$
$$= \frac{3\left(x^7y - \alpha\right)^2 + 4x\left(\alpha x^3 - y\right)^2}{x^7 - 1}$$
.

This function S is the integral along an extremal whose first end-point is $(1, \alpha)$ and whose second end-point is (x, y). By using the general variation formula which gives

$$\frac{\partial S}{\partial x} = F - y' F_{y'} = -H(x, y, p),$$
$$\frac{\partial S}{\partial y} = F_{y'} = p,$$

we conclude that

$$\frac{\partial S}{\partial x} + H\left(x, y, \frac{\partial S}{\partial y}\right) = 0.$$

Finally we have to check that the second-order derivative $\frac{\partial^2}{\partial y \partial \alpha} S$ is not identically zero. Differentiating S with respect to y yields

$$\frac{\partial S}{\partial y} = \frac{6x^7\left(x^7 y - \alpha\right) - 8x\left(\alpha x^3 - y\right)}{x^7 - 1}$$

and one more differentiation with respect to α yields

$$\frac{\partial^2 S}{\partial y \partial \alpha} = \frac{-6x^7 - 8x^4}{x^7 - 1} = \frac{-2x^4\left(3x^3 + 4\right)}{x^7 - 1}$$

which is not identically zero.

Problem 7. *(20 points)* Consider a string of constant tension T, constant linear density ρ, with both ends fixed at $x = 0$ and $x = \pi$, which vibrates freely so that the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is satisfied with $u(0, t) = u(\pi, t) = 0$ for $t \geq 0$, where $c^2 = \frac{T}{\rho}$ with $c > 0$ and where $u(x, t)$ is the vertical displacement of the string at position x and time t in its vibration. Assume that the vibration of the string starts with the initial position

$$u(x, 0) = \frac{\pi}{2} - \left|x - \frac{\pi}{2}\right| = \begin{cases} x \text{ for } 0 \leq x \leq \frac{\pi}{2} \\ \\ \pi - x \text{ for } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

and initial velocity $\frac{\partial u}{\partial t}(x, 0) \equiv 0$ for $0 \leq x \leq \pi$. Compute $u(x, t)$ for $0 \leq x \leq \pi$ and $t \geq 0$. (*Hint:* Compute the coefficients of an infinite sine series on $[0, \pi]$ as the Fourier series of its extension to $[-\pi, \pi]$ as an odd function.)

SOLUTION. First look for special product solutions $u(x,t) = X(x)T(t)$ which satisfy the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and the following initial and boundary conditions which specify zero values: $u(-\pi, t) = u(\pi, t) = 0$ for $t \geq 0$ and $\frac{\partial u}{\partial t}(x, 0) \equiv 0$ for $-\pi \leq x \leq \pi$. Then we have

$$\frac{\frac{d^2X}{dx^2}(x)}{X(x)} = \frac{\frac{1}{c^2}\frac{d^2T}{dt^2}(t)}{T(t)}$$

and as a consequence both sides must be equal to some constant $-\mu$. We now have the following two equations

$$\frac{d^2X}{dx^2}(x) + \mu X(x) = 0,$$
$$\frac{d^2T}{dt^2}(t) + c^2\mu T(t) = 0.$$

The boundary condition for $X(t)$ which specifies zero values now becomes $X(0) = X(\pi) = 0$ and the initial condition for $T(t)$ which specifies zero values now becomes $T'(0) = 0$. In order for the first ordinary differential equation

$$\frac{d^2X}{dx^2}(x) + \mu X(x) = 0$$

to admit a nontrivial solution $X(x)$ with the initial condition $X(0) = X(\pi) = 0$, we must have $\mu = n^2$ for some positive integer n and $X(x)$ must be equal to a nonzero constant multiple of $\sin nx$. The second ordinary differential equation

$$\frac{d^2T}{dt^2}(t) + c^2\mu T(t) = 0$$

with $\mu = n^2$ and the boundary condition $T'(0) = 0$ can be solved to give the conclusion that $T(t)$ must be equal to a nonzero constant multiple of $\cos(cnt)$. We now let

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos(cnt)$$

with undetermined coefficients b_n to fulfill the remaining initial condition

$$u(x,0)=\frac{\pi}{2}-\left|x-\frac{\pi}{2}\right|=\begin{cases} x \text{ for } 0\leq x\leq \frac{\pi}{2} \\ \pi-x \text{ for } \frac{\pi}{2}\leq x\leq \pi \end{cases}$$

by computing

$$b_n=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx dx+\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin nx dx$$

$$=\frac{2}{\pi}\left[\frac{x\cos nx}{n}\right]_{x=0}^{x=\frac{\pi}{2}}+\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin nx dx-\frac{2}{\pi}\left[\frac{(\pi-x)\cos nx}{n}\right]_{x=\frac{\pi}{2}}^{x=\pi}-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin nx dx$$

$$=\frac{4}{\pi n^2}\sin\frac{\pi n}{2},$$

which is nonzero only when n is an odd number. We use the index m with $n=2m+1$ to write the final answer as

$$u(x,t)=\sum_{m=0}^{\infty}(-1)^m\frac{4}{(2m+1)^2\pi}\sin\left((2m+1)x\right)\cos\left(c(2m+1)t\right).$$

Problem 8. Denote by $J_n(x)$ the Bessel function of order n defined by

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Fix $c>0$. Let α_j for positive integers j be the set of all positive zeroes x of $J_0(cx)$ arranged in increasing order. Assume as given the completeness of the orthogonal family of functions $J_0(\alpha_j x)$, indexed by the set of all positive integers j, with respect to the norm

$$f\mapsto\left(\int_0^c x\left|f(x)\right|^2dx\right)^{\frac{1}{2}}$$

(a) *(15 points)* Use the two identities

$$\frac{d}{dx}\left(xJ_1(x)\right)=xJ_0(x),\quad \frac{d}{dx}J_0(x)=-J_1(x)$$

and repeated integration by parts to show that there is a rational function $g(c,x)$ with universal coefficients such that

$$c^2 - x^2 = \sum_{j=1}^{\infty} g\left(c, \alpha_j\right) \frac{J_0\left(\alpha_j x\right)}{J_1\left(\alpha_j c\right)}.$$

Write down the rational function $g(c,x)$.

(b) *(15 points)* Let k be the thermal diffusivity. Consider the temperature distribution u on a disk of radius c centered at the origin such that the temperature at the boundary is always kept at 0 and the initial temperature distribution is $c^2 - r^2$ at the point of distance r from the center of the disk. In mathematical notations, the temperature $u(r,t)$ at time t and at distance r from the center of the disk satisfies (i) the partial differential equation

$$(\flat) \qquad \frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

for $0 < r < c$ and $t \geq 0$ and (ii) the boundary condition $u(c,t) = 0$ for $t \geq 0$ and (iii) the initial condition $u(r,0) = c^2 - r^2$. Solve the partial differential equation ($\flat$) by the method of the separation of variables and use Part (a) to express $u(r,t)$ in terms of $e^{-(\alpha_j)^2 kt}$, $J_0\left(\alpha_j r\right)$, c, α_j, and $J_1\left(\alpha_j c\right)$.

Solution. **(a)** First of all we use the chain rule to change

$$\frac{d}{dx}\left(xJ_1(x)\right) = xJ_0(x), \quad \frac{d}{dx}J_0(x) = -J_1(x)$$

to the identities

$$\frac{d}{dx}\left(xJ_1(\alpha_j x)\right) = \alpha_j x J_0(x), \quad \frac{d}{dx}J_0(\alpha x) = -\alpha_j J_1(\alpha_j x).$$

To compute the expansion of $c^2 - x^2$ in terms of the complete orthogonal family $J_0\left(\alpha_j x\right)$ indexed positive integers j, we first compute the inner product of the constant function 1 and the function $J_0\left(\alpha_j x\right)$ to get

$$\int_0^c xJ_0(\alpha_j x)dx = \frac{1}{\alpha_j}\int_0^c \left(xJ_1(\alpha_j x)\right)' dx = \left[\frac{x}{\alpha_j}J_1(\alpha_j x)\right]_{x=0}^{x=c} = \frac{c}{\alpha_j}\,J\left(\alpha_j c\right).$$

We now compute the inner product of the function $x \mapsto x^2$ and the function $J_0\left(\alpha_j x\right)$ to get

$$\int_0^c x^3 J_0(\alpha_j x)dx = \frac{1}{\alpha_j}\int_0^c x^2\left(xJ_1(\alpha_j x)\right)' dx$$

$$= \left[\frac{1}{\alpha_j} x^2 \left(x J_1(\alpha_j x) \right) \right]_{x=0}^{x=c} - \frac{2}{\alpha_j} \int_0^c x \left(x J_1(\alpha_j x) \right) dx$$

$$= \frac{c^3}{\alpha_j} J_1 \left(\alpha_j c \right) - \frac{2}{\alpha_j} \int_0^c x^2 J_1 \left(\alpha_j x \right) dx$$

$$= \frac{c^3}{\alpha_j} J_1 \left(\alpha_j c \right) + \frac{2}{\alpha_j^2} \int_0^c x^2 \left(J_0 \left(\alpha_j x \right) \right)' dx$$

$$= \frac{c^3}{\alpha_j} J_1 \left(\alpha_j c \right) + \left[\frac{2}{\alpha_j^2} x^2 J_0 \left(\alpha_j x \right) \right]_{x=0}^{x=c} - \frac{4}{\alpha_j^2} \int_0^c x J_0 \left(\alpha_j x \right) dx$$

$$= \frac{c^3}{\alpha_j} J_1 \left(\alpha_j c \right) - \frac{4c}{\alpha_j^3} J \left(\alpha_j c \right).$$

Putting the two computations together, we get

$$\int_0^c x \left(c^2 - x^2 \right) J_0(\alpha_j x) dx$$

$$= c^2 \frac{c}{\alpha_j} J_1 \left(\alpha_j c \right) - \left(\frac{c^3}{\alpha_j} J_1 \left(\alpha_j c \right) - \frac{4c}{\alpha_j^3} J \left(\alpha_j c \right) \right) = \frac{4c}{\alpha_j^3} J \left(\alpha_j c \right).$$

Now we use

$$\int_{x=0}^c x \left(J_0 \left(\alpha_j \, x \right) \right)^2 \, dx = \frac{c^2}{2} \left(J_1 \left(\alpha_j \, c \right) \right)^2$$

to get the expansion

$$c^2 - x^2 = \sum_{j=1}^{\infty} \frac{\int_0^c t \left(c^2 - t^2 \right) J_0(\alpha_j t) dt}{\int_{x=0}^c t \left(J_0 \left(\alpha_j \, t \right) \right)^2 dt} J_0 \left(\alpha_j x \right)$$

$$= \sum_{j=1}^{\infty} \frac{\frac{4c}{\alpha_j^3} J \left(\alpha_j c \right)}{\frac{c^2}{2} \left(J_1 \left(\alpha_j \, c \right) \right)^2} J_0 \left(\alpha_j x \right) = \sum_{j=1}^{\infty} \frac{8}{c \alpha_j^3} J_0 \left(\alpha_j x \right).$$

Thus the rational function $g(c, x)$ is given by

$$g(c, x) = \frac{8}{c x^3}$$

so that

$$c^2 - x^2 = \sum_{j=1}^{\infty} g \left(c, \alpha_j \right) \, J_0 \left(\alpha_j x \right).$$

(b) First look for special product solutions $u(x,t) = R(r)T(t)$ which satisfy the partial differential equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

and the following boundary condition which specifies zero values:

$$u(c,t) = 0 \quad \text{for} \quad t \geq 0.$$

Then we have

$$\frac{\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}}{R}(r) = \frac{\frac{1}{k}\frac{dT}{dt}(t)}{T(t)}$$

and as a consequence both sides must be equal to some constant $-\lambda$. We now have the following two equations

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda R = 0,$$
$$\frac{dT}{dt}(t) + k\lambda\mu T(t) = 0.$$

The boundary condition for $R(t)$ which specifies zero values now becomes $R(c) = 0$. By comparison with the rescaled Bessel differential equation

$$x^2y'' + xy' + \left(\alpha^2x^2 - n^2\right)y = 0$$

for the rescaled Bessel function $J_n(\alpha x)$ of order n, in order for the first ordinary differential equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda R = 0$$

to admit a nontrivial solution $R(r)$ with the initial condition $R(c) = 0$ (and $R(0)$ finite), we must have $R(r) = J_0(\alpha_j r)$ for some positive integer j and $\lambda = \alpha_j^2$. The second ordinary differential equation

$$\frac{dT}{dt}(t) + k\lambda T(t) = 0$$

with $\lambda = \alpha_j^2$ can be solved to give the conclusion that $T(t)$ must be equal to a nonzero constant multiple of $e^{-k\alpha_j^2 t}$. We now let

$$u(r,t) = \sum_{j=1}^{\infty} b_j J_0(\alpha_j r)\, e^{-k\alpha_j^2 t}$$

with undetermined coefficients b_j to fulfill the remaining initial condition $u(r,0) = c^2 - r^2$. From the result of Part (a) we have

$$c^2 - r^2 = \sum_{j=1}^{\infty} \frac{8}{c\alpha_j^3} J_0\left(\alpha_j r\right).$$

Hence we have $b_j = \frac{8}{c\alpha_j^3}$ and the final answer is

$$u(r,t) = \sum_{j=1}^{\infty} \frac{8}{c\alpha_j^3} J_0\left(\alpha_j r\right) e^{-k\alpha_j^2 t}.$$

Some Formulae

(1) Cauchy-Riemann equations for holomorphic function $u + iv$ of $x + iy$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Cauchy's integral formula for derivatives of holomorphic functions $f(z)$ of z is

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

(2) The Fourier series of $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx\right),$$

where the Fourier coefficients a_k $(0 \leq k < \infty)$ and b_k $(k \geq 1)$ are given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx,$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx.$$

Parseval's identity is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right).$$

(3) Poisson integral formulae for a harmonic function $u(z)$ on the unit disk:

$$u(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(\zeta) \operatorname{Re} \frac{\zeta + z}{\zeta - z} \, d\theta \quad \text{with} \quad \zeta = e^{i\theta} \quad \text{and} \quad |z| < 1.$$
$$u(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(\zeta) \, \frac{\zeta\bar{\zeta} - z\bar{z}}{|\zeta - z|^2} \, d\theta \quad \text{with} \quad \zeta = e^{i\theta} \quad \text{and} \quad |z| < 1.$$

$$u\left(re^{i\theta}\right)=\frac{1}{2\pi}\int_{\varphi=0}^{2\pi}\frac{1-r^2}{1+r^2-2r\cos\left(\varphi-\theta\right)}\,u\left(e^{i\varphi}\right)\,d\varphi\quad\text{for}\quad 0\leq r<1.$$

The harmonic measure $\omega_{\alpha,\beta}$ is a function on the unit disk characterized by the following properties.

(i) $\omega_{\alpha,\beta}$ is harmonic on the open unit disk,

(ii) $\omega_{\alpha,\beta}$ assumes the value 1 on the arc in the counterclockwise sense between $e^{i\alpha}$ and $e^{i\beta}$, and

(iii) $\omega_{\alpha,\beta}$ assumes the value 0 on the arc in the counterclockwise sense between $e^{i\beta}$ and $e^{i\alpha}$.

Its formula is

$$\omega_{\alpha,\beta}(z)=\frac{1}{\pi}\left(\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right)-\frac{1}{2}\left(\beta-\alpha\right)\right),$$

where the argument function is defined with cuts

$$\left\{re^{i\alpha}\,\middle|\,r\geq 1\right\},\quad\left\{re^{i\beta}\,\middle|\,r\geq 1\right\}.$$

(4) For a 2-dimensional steady irrotational incompressible fluid flow of constant density whose complex potential function is $F(x+iy)=\varphi(x,y)+i\psi(x,y)$, the velocity as a complex number at the point z is given by the complex conjugate $\overline{F'(z)}$ of the derivative $F'(z)$ of the holomorphic function $F(z)$ and the streamlines are given by ψ being constant. For a source bounded by the streamline $\psi=\alpha$ on the right and $\psi=\beta$ on the left the magnitude of the source is $\beta-\alpha$.

(5) Euler-Lagrange equation

$$F_y\left(x,y(x),y'(x)\right)-\frac{d}{dx}F_{y'}\left(x,y(x),y'(x)\right)\equiv 0$$

on $[x_1,x_2]$ for the functional

$$J[y]=\int_{x=x_1}^{x_2}F\left(x,y(x),y'(x)\right)\,dx.$$

(6) First integral

$$F - y' F_{y'} = \text{constant}$$

of the Euler-Lagrange equation for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y, y'\right)\, dx$$

when $F(x, y, y')$ is independent of x.

(7) General variation formula

$$\delta J = \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'} \right) (\partial_t y)\, dx + F_{y'} \delta y \Big|_{x=x_1}^{x=x_2} + \left(F - y' F_{y'}\right) \delta x \Big|_{x=x_1}^{x=x_2}$$

for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y, y'\right)\, dx.$$

(8) Weierstrass-Erdmann corner condition

$$\left(F_{y'} \Big|_{x=\xi-0}^{x=\xi+0} \right) \left(\delta y \Big|_{x=\xi} \right) + \left(\left(F - y' F_{y'}\right) \Big|_{x=\xi+0}^{x=\xi-0} \right) \left(\delta x \Big|_{x=\xi} \right) = 0$$

for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y, y'\right)\, dx$$

when there is a corner at $x = \xi$ with $x_1 < \xi < x_2$ for the extremal.

(9) Euler-Lagrange equation

$$\left(F - \lambda G\right)_y - \frac{d}{dx} \left(F - \lambda G\right)_{y'} = 0$$

with Lagrange multiplier $\lambda \in \mathbb{R}$ for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y, y'\right)\, dx$$

subject to the integral constraint

$$\int_{x=x_1}^{x_2} G\left(x, y, y'\right)\, dx = \ell \in \mathbb{R}.$$

(10) Euler-Lagrange equation

$$(F-\lambda(x)g)_y - \frac{d}{dx}(F-\lambda(x)g)_{y'} = 0,$$
$$(F-\lambda(x)g)_z - \frac{d}{dx}(F-\lambda(x)g)_{z'} = 0$$

with Lagrange multiplier $\lambda(x)$ (which is a real-valued function of x) for the functional

$$J[y,z] = \int_{x=x_1}^{x_2} F(x,y,z,y',z')\,dx$$

subject to the pointwise constraint

$$g(x,y,z) = 0.$$

(11) The canonical variables x, y, p and the Hamiltonian H for the functional

$$J[y] = \int_{x=x_1}^{x_2} F(x,y,y')\,dx$$

are given by $p = F_{y'}$ and $H = -F + y'p$. The canonical differential equations are

$$\frac{dy}{dx} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dx} = -\frac{\partial H}{\partial y}.$$

(12) The action integral for the functional

$$J[y] = \int_{x=x_1}^{x_2} F(x,y,y')\,dx$$

is the integral

$$\int_{x_1}^{x_2} F(x,y,y')\,dx$$

computed along an extremal.

(13) The canonical transformation defined by the generating function $\Phi(x,y,Y)$ is given by

$$p = \frac{\partial \Phi}{\partial y}, \quad P = -\frac{\partial \Phi}{\partial Y}, \quad H^* = H + \frac{\partial \Phi}{\partial x}.$$

The canonical transformation defined by the generating function $\Psi\left(x,y,P\right)$ is given by

$$p=\frac{\partial\Psi}{\partial y},\quad Y=\frac{\partial\Psi}{\partial P},\quad H^{*}=H+\frac{\partial\Psi}{\partial x}.$$

(14) Legendre's necessary condition for a local minimum for the variational problem of the functional

$$J[y]=\int_{x=x_1}^{x_2}F\left(x,y,y'\right)\,dx$$

is $P(x)\geq 0$ and the Jacobi differential equation is

$$-\frac{d}{dx}\left(Ph'\right)+Qh=0$$

with unknown function $h=h(x)$, where

$$P(x)=\frac{1}{2}F_{y'y'},\qquad Q(x)=\frac{1}{2}\left(F_{yy}-\left(\frac{d}{dx}F_{yy'}\right)\right).$$

(15) The Bessel function

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

of order n satisfies the differential equation

$$x^2y''+xy'+\left(x^2-n^2\right)y=0$$

and is given alternatively by the generating function

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}J_n(x)t^n.$$

The rescaled Bessel function $J_n\left(\alpha x\right)$ satisfies the rescaled Bessel differential equation

$$x^2y''+xy'+\left(\alpha^2x^2-n^2\right)y=0$$

The derivatives of Bessel functions are given by

$$\left(\frac{J_n}{x^n}\right)'=-\frac{J_{n+1}}{x^n},$$

$$(x^n J_n)' = x^n J_{n-1}$$

and their recurrent relation is

$$x J_{n+1}(x) = 2n J_n(x) - x J_{n-1}(x).$$

For fixed n and fixed $c > 0$ let $\alpha_{n,\ell}$, indexed by the set of all positive integers ℓ, be the set of all nonnegative roots x of $J_n(x\,c) = 0$. The weighted square integrals of rescaled Bessel functions are given by

$$\int_{x=0}^{c} x\left(J_n\left(\alpha_{n,\ell}\,x\right)\right)^2\,dx = \frac{c^2}{2}\left(J_n'\left(\alpha_{n,\ell}\,c\right)\right)^2 = \frac{c^2}{2}\left(J_{n+1}\left(\alpha_{n,\ell}\,c\right)\right)^2.$$