Solutions to Problems in
Math 115 Second Mid-Term Examination
December 14, 2006, 11:30 a.m. - 1 p.m.
Science Center 411

Some formulae are provided at the end of the problem list. Not all of them are necessary for the test.

Problem 1. Consider the Poisson integral formula

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}\, u\left(e^{i\varphi}\right)\, d\varphi,$$

which expresses the value $u\left(re^{i\theta}\right)$ of a harmonic function u at the point $re^{i\theta}$ (for $0 \le r < 1$) in terms of its values $u\left(e^{i\varphi}\right)$ at points $e^{i\varphi}$ of the unit circle.

(a) *(12%)* Use the above Poisson integral formula to show that

$$\int_{\theta=0}^{2\pi} \frac{d\theta}{1+\tau^2-2\tau\cos\theta} = \frac{2\pi}{1-\tau^2}$$

for $0 < \tau < 1$.

(b) *(13%)* Let $0 < \tau < 1$ and $0 \le \alpha < \frac{\pi}{2}$ and $\alpha < \beta < \frac{\pi}{2}$. Express

$$\int_{\theta=\alpha}^{\beta} \frac{d\theta}{1+\tau^2-2\tau\cos\theta}$$

in terms of τ, π, α, β, rational functions, trigonometric functions, and inverses of trigonometric functions.

Hint: Use the argument function to write down the harmonic measure for the counterclockwise arc from $e^{i\alpha}$ to $e^{i\beta}$ (which is the harmonic function on the unit disk assuming boundary value 1 on the counterclockwise arc from $e^{i\alpha}$ to $e^{i\beta}$ and boundary value 0 on the counterclockwise arc from $e^{i\beta}$ to $e^{i\alpha}$).

Solution.

(a) By applying the Poisson integral formula to $u \equiv 1$ at the point $z = \tau$ (with $r = \tau$ and $\theta = 0$), we get

$$1 = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \frac{1-\tau^2}{1+\tau^2-2\tau\cos\varphi}\, d\varphi,$$

which is the same as

$$\int_{\theta=0}^{2\pi} \frac{d\theta}{1+\tau^2-2\tau\cos\theta} = \frac{2\pi}{1-\tau^2}$$

after we change the dummy variable φ to θ.

(b) Let $\omega_{\alpha,\beta}$ be the harmonic measure which is a function on the unit disk characterized by the following properties.

(i) $\omega_{\alpha,\beta}$ is harmonic on the open unit disk,

(ii) $\omega_{\alpha,\beta}$ assumes the value 1 on the arc in the counterclockwise sense between $e^{i\alpha}$ and $e^{i\beta}$, and

(iii) $\omega_{\alpha,\beta}$ assumes the value 0 on the arc in the counterclockwise sense between $e^{i\beta}$ and $e^{i\alpha}$.

When $z = e^{i\theta}$ is on the arc between $e^{i\beta}$ and $e^{i\alpha}$ in the counterclockwise, we have

$$\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right) \equiv \frac{1}{2}(\beta-\alpha),$$

where the argument function is defined with cuts

$$\left\{ re^{i\alpha} \,\middle|\, r \geq 1 \right\}, \quad \left\{ re^{i\beta} \,\middle|\, r \geq 1 \right\}.$$

When $z = e^{i\theta}$ is on the arc between $e^{i\alpha}$ and $e^{i\beta}$ in the counterclockwise sense so that $\alpha < \theta < \beta$, we have

$$\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right) = \pi + \frac{1}{2}(\beta-\alpha).$$

We conclude that

$$\omega_{\alpha,\beta} = \frac{1}{\pi}\left(\arg\left(\frac{z-e^{i\beta}}{z-e^{i\alpha}}\right) - \frac{1}{2}(\beta-\alpha)\right).$$

When $z = \tau$, we have

$$\omega_{\alpha,\beta}(z)\bigg|_{z=r} = \frac{1}{\pi}\left(\arg\left(\frac{\tau-e^{i\beta}}{\tau-e^{i\alpha}}\right) - \frac{1}{2}(\beta-\alpha)\right)$$

$$= \frac{1}{\pi}\left(\arg\left(\tau - e^{i\beta}\right) - \arg\left(\tau - e^{i\alpha}\right) - \frac{1}{2}(\beta - \alpha)\right)$$

$$= \frac{1}{\pi}\left(\tan^{-1}\left(\frac{\sin\beta}{\cos\beta - \tau}\right) - \tan^{-1}\left(\frac{\sin\alpha}{\cos\alpha - \tau}\right) - \frac{1}{2}(\beta - \alpha)\right),$$

where the range of $\tan^{-1}$ is in $[0, \pi]$. On the other hand, by Poisson's integral formula we have

$$\omega_{\alpha,\beta}(z)\bigg|_{z=\tau} = \frac{1}{2\pi}\int_{\varphi=\alpha}^{\beta} \frac{1-\tau^2}{1+\tau^2 - 2\tau\cos\theta}\, d\theta.$$

Thus we conclude that

$$\int_{\theta=0}^{2\pi} \frac{d\theta}{1+\tau^2 - 2\tau\cos\theta} = \frac{2\pi}{1-\tau^2}\,\omega_{\alpha,\beta}(z)\bigg|_{z=\tau}$$

$$= \frac{2}{1-\tau^2}\left(\tan^{-1}\left(\frac{\sin\beta}{\cos\beta - \tau}\right) - \tan^{-1}\left(\frac{\sin\alpha}{\cos\alpha - \tau}\right) - \frac{1}{2}(\beta - \alpha)\right)$$

with the range of $\tan^{-1}$ in $[0, \pi]$.

Problem 2. Consider the functional

$$(\dagger) \qquad J[y] = \int_{x_1}^{x_2} F\left(x, y, y'\right)$$

with

$$F\left(x, y, y'\right) = y^2 + y'^2 + 2ye^{2x}.$$

Let $y_1 = y(x_1)$ and $y_2 = y(x_2)$.

(a) *(12%)* Write down the Euler-Lagrange equation for the variation of the functional $J[y]$ and solve it to get a function $y = y(x)$ of the variable x with two yet-to-be-determined constants a and b in it.

(b) *(12%)* For the case where $x_1 = 0$, $x_2 = 1$, $y_1 = 1$, and $y_2 = \frac{1}{3}\left(e^2 + e + e^{-1}\right)$, compute the extremal function $y = y(x)$ for the functional $J[y]$ with the two given end-points (x_1, y_1) and (x_2, y_2).

(c) *(12%)* The the case where $x_1 = 0$, $x_2 = 1$, and $y_1 = 1$, and y_2 is allowed to vary freely, compute the extremal function $y = y(x)$ for the functional $J[y]$.

(d) *(13%)* For the case where $x_1 = 0$, $x_2 = 1$, $y_1 = 1$, and $y_2 = \frac{1}{3}\left(e^2 + e + e^{-1}\right)$, compute the extremal function $y = y(x)$ for the functional $J[y]$ with the two given end-points (x_1, y_1) and (x_2, y_2), subject to the integral constraint

$$\int_{x_1}^{x_2} 2y\, dx = 1.$$

(e) *(13%)* Show that the extremal $y = y(x)$ for $J[y]$ obtained in Part (b) with fixed end-points $(0, 1)$ and $\left(1, \frac{1}{3}\left(e^2 + e + e^{-1}\right)\right)$ is a local (weak) minimum by verifying that for the Jacobi equation of the functional $J[y]$ there is no point in $(0, 1]$ conjugate to the point 0 and that the strengthened Legendre condition is satisfied.

(f) *(13%)* Write down the Hamiltonian

$$H(x, y, p) = y'p - F(x, y, y')$$

for the functional $J[y]$ of (†), where $p = F_{y'}$. Use the action integral to find explicitly a solution $S(x, y, \alpha)$ of the Hamilton-Jacobi differential equation

$$\frac{\partial S}{\partial x} + H\left(x, y, \frac{\partial S}{\partial y}\right) = 0,$$

where α is a constant parameter, such that

$$\frac{\partial^2 S}{\partial y \partial \alpha}$$

is not identically zero.

Solution.

(a) The Euler-Lagrange equation of the functional $J]y]$ of (†) is

$$F_y - \frac{d}{dx} F_{y'} = 0,$$

which for our case of

$$F(x, y, y') = y^2 + y'^2 + 2ye^{2x}$$

becomes

$$\frac{\partial}{\partial y}\left(y^2 + y'^2 + 2ye^{2x}\right) - \frac{d}{dx}\frac{\partial}{\partial y'}\left(y^2 + y'^2 + 2ye^{2x}\right) = 0,$$

whose expansion is

$$2y + 2e^{2x} - 2y'' = 0$$

or

$$(*) \qquad y'' - y = e^{2x}.$$

The solution of the homogeneous part $y'' - y = 0$ of $(*)$ is obtained by determining the unknown constant r in $y = e^{rx}$ by substitution into $y'' - y = 0$, which yields $r^2 - 1 = 0$ and $r = \pm 1$. Thus $y = ae^x + be^{-x}$ with two yet-to-be-determined constants is a solution of the homogeneous part $y'' - y = 0$ of $(*)$. A particular solution of $(*)$ is of the form Ce^{2x} with the constant C determined by

$$4Ce^{2x} - Ce^{2x} = e^{2x}$$

so that $C = \frac{1}{3}$. Thus

$$y = ae^x + be^{-x} + \frac{1}{3}e^{2x}$$

is the general solution of the Euler-Lagrange equation for the functional $J[y]$ of (†).

(b) We determine the constants a and b of the solution

$$y = ae^x + be^{-x} + \frac{1}{3}e^{2x}$$

of the Euler-Lagrange equation for the functional $J[y]$ of (†) given in Part (a) from the condition

$$y(0) = 1, \qquad y(1) = \frac{1}{3}\left(e^2 + e + e^{-1}\right),$$

which means

$$\begin{cases} a + b + \frac{1}{3} = 1 \\ ae + be^{-1} + \frac{1}{3}e^2 = \frac{1}{3}\left(e^2 + e + e^{-1}\right), \end{cases}$$

yielding $a = b = \frac{1}{3}$. Thus the answer is

$$y = \frac{1}{3}\left(e^x + e^{-x} + e^{2x}\right).$$

(c) We apply the general variation formula

$$\delta J = \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'} \right) (\partial_t y)\, dx + F_{y'} \delta y \Big|_{x=x_1}^{x=x_2} + (F - y' F_{y'})\, \delta x \Big|_{x=x_1}^{x=x_2}$$

for the functional $J[y]$ of (†). Since the Euler-Lagrange equation $F_y - \frac{d}{dx} F_{y'} = 0$ is satisfied and

$$\delta x \Big|_{x=x_1} = 0, \quad \delta y \Big|_{x=x_1} = 0, \quad \delta x \Big|_{x=x_2} = 0,$$

it follows that

$$\delta J = F_{y'} \delta y \Big|_{x=x_2}.$$

From $\delta J = 0$ and the arbitrariness of $\delta y \Big|_{x=x_2}$ we conclude that $F_{y'} = 0$ at $x = x_2$. Since

$$F_{y'} = 2y' = \frac{2}{3} e^{2x} + a\, e^{x} - b\, e^{-x}$$

which is equal to $\frac{2}{3} e^2 + a\, e - b\, e^{-1}$ at $x = 1$, it follows that we have the following condition for the yet-to-be-determined constants a and b.

$$\begin{cases} a + b + \frac{1}{3} = 1 \\ \frac{2}{3} e^2 + a\, e - b\, e^{-1} = 0, \end{cases}$$

yielding

$$a = \frac{2\,(e^{-1} - e^2)}{3\,(e + e^{-1})}, \quad b = \frac{2\,(e + e^2)}{3\,(e + e^{-1})},$$

when we add e^{-1} times the first equation to the second equation to get

$$\frac{2}{3} e^2 - \frac{2}{3} e^{-1} + a \left(e + e^{-1} \right) = 0$$

to solve for a and then use the first equation to get $b = \frac{2}{3} - a$ to solve for b. So the final answer is

$$y = \frac{2\,(e^{-1} - e^2)}{3\,(e + e^{-1})}\, e^{x} + \frac{2\,(e + e^2)}{3\,(e + e^{-1})}\, e^{-x} + \frac{1}{3}\, e^{2x}.$$

(d) We let $\ell = 1$ and $G(x, y, y') = 2y$ and apply the Euler-Lagrange equation

$$(F - \lambda G)_y - \frac{d}{dx}(F - \lambda G)_{y'} = 0$$

with Lagrange multiplier $\lambda \in \mathbb{R}$ for the functional $J[y]$ of (†) subject to the integral constraint

$$\int_{x=x_1}^{x_2} G(x, y, y')\, dx = \ell \in \mathbb{R}.$$

Since $F - \lambda G = y^2 + y'^2 + 2y\,(e^{2x} - \lambda)$, the Euler-Lagrange equation becomes $2y + 2\,e^{2x} - 2\lambda - 2y'' = 0$ or $y'' - y = e^{2x} - \lambda$. So the general solution is

$$y = ae^x + be^{-x} + \frac{1}{3}e^{2x} + \lambda.$$

We have the following three conditions (which are three linear equations)

$$\begin{cases} a + b + \frac{1}{3} + \lambda = 1 \\ ae + be^{-1} + \frac{1}{3}e^2 + \lambda = \frac{1}{3}\left(e^2 + e + e^{-1}\right) \\ \int_0^1 \left(ae^x + be^{-x} + \frac{1}{3}e^{2x} + \lambda\right) dx = \frac{1}{2} \end{cases}$$

to determine the three constants a, b, and λ. After performing the integration in the third equation, we obtain the following three conditions

$$\begin{cases} a + b + \lambda = \frac{2}{3} \\ ae + be^{-1} + \lambda = \frac{1}{3}\left(e + e^{-1}\right) \\ a\,(e - 1) + b\,(1 - e^{-1}) + \lambda = \frac{2}{3} - \frac{1}{6}\,e^2 \end{cases}$$

which, after subtracting the first equation from the second and the third equations, become

$$\begin{cases} a\,(e - 1) + b\,(e^{-1} - 1) = \frac{1}{3}\left(e - 2 + e^{-1}\right) \\ a\,(e - 2) - b\,e^{-1} = -\frac{1}{6}\,e^2 \end{cases}$$

whose solutions a and b by Cramer's rule are

$$a = \frac{\begin{vmatrix} \frac{1}{3}\left(e - 2 + e^{-1}\right) & e^{-1} - 1 \\ -\frac{1}{6}\,e^2 & -e^{-1} \end{vmatrix}}{\begin{vmatrix} e - 1 & e^{-1} - 1 \\ e - 2 & -e^{-1} \end{vmatrix}},$$

$$b = \frac{\begin{vmatrix} e-1 & \frac{1}{3}\left(e-2+e^{-1}\right) \\ e-2 & -\frac{1}{6}\,e^2 \end{vmatrix}}{\begin{vmatrix} e-1 & e^{-1}-1 \\ e-2 & -e^{-1} \end{vmatrix}}.$$

Computing the determinants, we get

$$a = \frac{-\frac{1}{6}e^2+\frac{1}{6}e-\frac{1}{3}+\frac{2}{3}e^{-1}-\frac{1}{3}e^{-2}}{e-4+3e^{-1}},$$

$$b = \frac{-\frac{1}{6}e^3-\frac{1}{6}e^2+\frac{4}{3}e-\frac{5}{3}+\frac{2}{3}e^{-1}}{e-4+3e^{-1}}$$

and

$$\lambda = \frac{2}{3}-a-b$$

$$= \frac{2}{3} - \frac{-\frac{1}{6}e^2+\frac{1}{6}e-\frac{1}{3}+\frac{2}{3}e^{-1}-\frac{1}{3}e^{-2}}{e-4+3e^{-1}} - \frac{-\frac{1}{6}e^3-\frac{1}{6}e^2+\frac{4}{3}e-\frac{5}{3}+\frac{2}{3}e^{-1}}{e-4+3e^{-1}}$$

$$= \frac{2\left(e-4+3e^{-1}\right)+\frac{1}{2}e^2-\frac{1}{2}e+1-2e^{-1}+e^{-2}+\frac{1}{2}e^3+\frac{1}{2}e^2-4e+5-2e^{-1}}{3\left(e-4+3e^{-1}\right)}$$

$$= \frac{\frac{1}{2}\,e^3+e^2-\frac{5}{2}\,e-2+2\,e^{-1}+e^{-2}}{3\left(e-4+3e^{-1}\right)}.$$

(e) We compute

$$P(x) = \frac{1}{2}F_{y'y'}, \qquad Q(x) = \frac{1}{2}\left(F_{yy}-\left(\frac{d}{dx}F_{yy'}\right)\right)$$

and get

$$P(x) \equiv 1, \qquad Q(x) \equiv 1$$

so that the Jacobi differential equation

$$-\frac{d}{dx}\left(Ph'\right)+Qh=0$$

for the unknown function $h(x)$ on $[0,1]$ is simply

$$h''-h=0$$

For a solution with $h(0)=0$ and $h'(0)\neq 0$ we can take $h(x)=e^x-e^{-x}$, which is nonzero for $0<x\leq 1$. Thus for the Jacobi equation of the functional $J[y]$ there is no point in $(0,1]$ conjugate to the point 0. Since $P>0$ on $[0,1]$, the strengthened Legendre condition is satisfied and we conclude that the extremal $y=y(x)$ for $J[y]$ obtained in Part (b) with fixed end-points $(0,1)$ and $\left(1,\frac{1}{3}\left(e^2+e+e^{-1}\right)\right)$ is a local (weak) minimum.

(f) Using $p = F_{y'} = 2y'$, we compute the Hamiltonian $H = y'p - F$ to be

$$H = \frac{p^2}{2} - \left(\frac{p^2}{4} + y^2 + 2ye^{2x}\right) = \frac{p^2}{4} - y^2 - 2ye^{2x}.$$

Introduce the action integral

$$S = \int_{x=x_1}^{x_2} F\left(x, y, y'\right) dx$$

along an extremal $y = y(x)$ with end-points (x_1, y_1) and (x_2, y_2), as a function of x_1, y_1, x_2, y_2. An extremal is necessarily of the form

$$y = ae^x + be^{-x} + \frac{1}{3}e^{2x}$$

with $a = a\left(x_1, y_1, x_2, y_2\right)$ and $b = b\left(x_1, y_1, x_2, y_2\right)$ depending on x_1, y_1, x_2, y_2. We set $x_1 = 0$ and $y_1 = \alpha$. We use $y_2 = y(x_2)$ to get

$$(\ddagger) \qquad \alpha = a + b + \frac{1}{3} \quad \text{and} \quad y_2 = ae^{x_2} + be^{-x_2} + \frac{1}{3}e^{2x_2}.$$

From

$$y' = ae^x - be^{-x} + \frac{2}{3}e^{2x}$$

it follows that

$$\begin{aligned}
F\left(x, y, y'\right) &= y'^2 + y^2 + 2ye^{2x} \\
&= \left(ae^x - be^{-x} + \frac{2}{3}e^{2x}\right)^2 + \left(ae^x + be^{-x} + \frac{1}{3}e^{2x}\right)\left(ae^x + be^{-x} + \frac{7}{3}e^{2x}\right) \\
&= \left(a^2e^{2x} + b^2e^{-2x} + \frac{4}{9}e^{4x} - 2ab + \frac{4}{3}ae^{3x} - \frac{4}{3}be^x\right) \\
&+ \left(a^2e^{2x} + ab + \frac{7}{3}ae^{3x} + ba + b^2e^{-2x} + \frac{7}{3}be^x + \frac{1}{3}ae^{3x} + \frac{1}{3}be^x + \frac{7}{9}e^{4x}\right) \\
&= \frac{11}{9}e^{4x} + 4ae^{3x} + 2a^2e^{2x} + \frac{4}{3}be^x - ab + 2b^2e^{-2x}
\end{aligned}$$

and

$$S = \int_{x=x_1}^{x_2} \left(\frac{11}{9}e^{4x} + 4ae^{3x} + 2a^2e^{2x} + \frac{4}{3}be^x - ab + 2b^2e^{-2x}\right) dx$$

$$
\begin{aligned}
&= \frac{11}{36}e^{4x} + \frac{4}{3}ae^{3x} + a^2e^{2x} + \frac{4}{3}be^{x} - abx - b^2e^{-2x}\Big|_{x=x_1}^{x=x_2} \\
&= \frac{11}{36}\left(e^{4x_2}-1\right) + \frac{4}{3}a\left(e^{3x_2}-1\right) + a^2\left(e^{2x_2}-1\right) \\
&\quad + \frac{4}{3}b\left(e^{x_2}-1\right) - abx_2 - b^2\left(e^{-2x_2}-1\right).
\end{aligned}
$$

We now solve for a and b in (‡) in terms of α, x_2, y_2 to get

$$
\begin{aligned}
a &= \frac{3\left(y_2-\alpha e^{-x_2}\right)+e^{-x_2}-e^{2x_2}}{e^{x_2}-e^{-x_2}}, \\
b &= \frac{3\left(\alpha e^{x_2}-y_2\right)+e^{2x_2}-e^{-x_2}}{e^{x_2}-e^{-x_2}}
\end{aligned}
$$

We put the values of a and b into S to get

$$
\begin{aligned}
S = \frac{11}{36}\left(e^{4x_2}-1\right) &+ \frac{4}{3}\left(\frac{3\left(y_2-\alpha e^{-x_2}\right)+e^{-x_2}-e^{2x_2}}{e^{x_2}-e^{-x_2}}\right)\left(e^{3x_2}-1\right) \\
&+ \left(\frac{3\left(y_2-\alpha e^{-x_2}\right)+e^{-x_2}-e^{2x_2}}{e^{x_2}-e^{-x_2}}\right)^2\left(e^{2x_2}-1\right) \\
&+ \frac{4}{3}\left(\frac{3\left(\alpha e^{x_2}-y_2\right)+e^{2x_2}-e^{-x_2}}{e^{x_2}-e^{-x_2}}\right)\left(e^{x_2}-1\right) \\
&- \left(\frac{3\left(y_2-\alpha e^{-x_2}\right)+e^{-x_2}-e^{2x_2}}{e^{x_2}-e^{-x_2}}\right)\left(\frac{3\left(\alpha e^{x_2}-y_2\right)+e^{2x_2}-e^{-x_2}}{e^{x_2}-e^{-x_2}}\right)x_2 \\
&- \left(\frac{3\left(\alpha e^{x_2}-y_2\right)+e^{2x_2}-e^{-x_2}}{e^{x_2}-e^{-x_2}}\right)^2\left(e^{-2x_2}-1\right).
\end{aligned}
$$

We now replace x_2 by x and y_2 by y to get the final answer

$$
\begin{aligned}
S = \frac{11}{36}\left(e^{4x}-1\right) &+ \frac{4}{3}\left(\frac{3\left(y-\alpha e^{-x}\right)+e^{-x}-e^{2x}}{e^{x}-e^{-x}}\right)\left(e^{3x}-1\right) \\
+ \left(\frac{3\left(y-\alpha e^{-x}\right)+e^{-x}-e^{2x}}{e^{x}-e^{-x}}\right)^2&\left(e^{2x}-1\right) + \frac{4}{3}\left(\frac{3\left(\alpha e^{x}-y\right)+e^{2x}-e^{-x}}{e^{x}-e^{-x}}\right)\left(e^{x}-1\right) \\
&- \left(\frac{3\left(y-\alpha e^{-x}\right)+e^{-x}-e^{2x}}{e^{x}-e^{-x}}\right)\left(\frac{3\left(\alpha e^{x}-y\right)+e^{2x}-e^{-x}}{e^{x}-e^{-x}}\right)x \\
&- \left(\frac{3\left(\alpha e^{x}-y\right)+e^{2x}-e^{-x}}{e^{x}-e^{-x}}\right)^2\left(e^{-2x}-1\right).
\end{aligned}
$$

This function S is the integral along an extremal whose first end-point is $(0,\alpha)$ and whose second end-point is (x,y). By using the general variation formula which gives

$$\frac{\partial S}{\partial x} = F - y' F_{y'} = -H\left(x,y,p\right),$$
$$\frac{\partial S}{\partial y} = F_{y'} = p,$$

we conclude that

$$\frac{\partial S}{\partial x} + H\left(x,y,\frac{\partial S}{\partial y}\right) = 0.$$

Finally we have to check that the second-order derivative $\frac{\partial^2}{\partial y \partial \alpha} S$ is not identically zero. Differentiating S with respect to y yields

$$\begin{gathered}\frac{\partial S}{\partial y} = \frac{4}{3}\left(\frac{3}{e^x - e^{-x}}\right)\left(e^{3x}-1\right)\\ +\frac{6}{e^x - e^{-x}}\left(\frac{3\left(y-\alpha e^{-x}\right)+e^{-x}-e^{2x}}{e^x - e^{-x}}\right)\left(e^{2x}-1\right)+\left(\frac{-4}{e^x - e^{x}}\right)\left(e^{x}-1\right)\\ -\left(\frac{3}{e^x - e^{-x}}\right)\left(\frac{3\left(\alpha e^{x}-y\right)+e^{2x}-e^{-x}}{e^x - e^{-x}}\right)x-\left(\frac{3\left(y-\alpha e^{-x}\right)+e^{-x}-e^{2x}}{e^x - e^{-x}}\right)\left(\frac{-3}{e^x - e^{-x}}\right)x\\ -\frac{-6}{e^x - e^{-x}}\left(\frac{3\left(\alpha e^{x}-y\right)+e^{2x}-e^{-x}}{e^x - e^{-x}}\right)\left(e^{-2x}-1\right).\end{gathered}$$

and one more differentiation with respect to α yields

$$\begin{gathered}\frac{\partial^2 S}{\partial y \partial \alpha} = \frac{6}{e^x - e^{-x}}\left(\frac{-3e^{-x}}{e^x - e^{-x}}\right)\left(e^{2x}-1\right)\\ -\left(\frac{3}{e^x - e^{-x}}\right)\left(\frac{3e^{x}}{e^x - e^{-x}}\right)x-\left(\frac{-3e^{-x}}{e^x - e^{-x}}\right)\left(\frac{-3}{e^x - e^{-x}}\right)x\\ -\frac{-6}{e^x - e^{-x}}\left(\frac{3e^{x}}{e^x - e^{-x}}\right)\left(e^{-2x}-1\right)\\ = -\frac{36}{e^x - e^{-x}}+\frac{18e^{x}x}{\left(c^x - e^{-x}\right)^2}.\end{gathered}$$

which is not identically zero.

Some Formulae

(1) Poisson integral formulae for a harmonic function $u(z)$ on the unit disk:

$$u(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(\zeta)\,\mathrm{Re}\,\frac{\zeta+z}{\zeta-z}\,d\theta \quad \text{with} \quad \zeta = e^{i\theta} \quad \text{and} \quad |z|<1.$$

$$u(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(\zeta)\,\frac{\zeta\bar{\zeta}-z\bar{z}}{|\zeta-z|^2}\,d\theta \quad \text{with} \quad \zeta = e^{i\theta} \quad \text{and} \quad |z|<1.$$

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}\,u\left(e^{i\varphi}\right)\,d\varphi \quad \text{for} \quad 0\le r<1.$$

(2) Euler-Lagrange equation

$$F_y\left(x, y(x), y'(x)\right) - \frac{d}{dx}F_{y'}\left(x, y(x), y'(x)\right) \equiv 0$$

on $[x_1, x_2]$ for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y(x), y'(x)\right)\,dx.$$

(3) First integral

$$F - y'F_{y'} = \text{constant}$$

of the Euler-Lagrange equation for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y, y'\right)\,dx$$

when $F(x, y, y')$ is independent of x.

(4) General variation formula

$$\delta J = \int_{x_1}^{x_2}\left(F_y - \frac{d}{dx}F_{y'}\right)(\partial_t y)\,dx + F_{y'}\delta y\Bigg|_{x=x_1}^{x=x_2} + \left(F - y'F_{y'}\right)\delta x\Bigg|_{x=x_1}^{x=x_2}$$

for the functional

$$J[y] = \int_{x=x_1}^{x_2} F\left(x, y, y'\right)\,dx.$$

(5) Weierstrass-Erdmann corner condition

$$\left(F_{y'}\Big|_{x=\xi-0}^{x=\xi+0}\right)\left(\delta y\Big|_{x=\xi}\right)+\left((F-y'F_{y'})\Big|_{x=\xi+0}^{x=\xi-0}\right)\left(\delta x\Big|_{x=\xi}\right)=0$$

for the functional

$$J[y]=\int_{x=x_1}^{x_2} F\left(x,y,y'\right)\,dx$$

when there is a corner at $x=\xi$ with $x_1<\xi<x_2$ for the extremal.

(6) Euler-Lagrange equation

$$(F-\lambda G)_y-\frac{d}{dx}(F-\lambda G)_{y'}=0$$

with Lagrange multiplier $\lambda\in\mathbb{R}$ for the functional

$$J[y]=\int_{x=x_1}^{x_2} F\left(x,y,y'\right)\,dx$$

subject to the integral constraint

$$\int_{x=x_1}^{x_2} G\left(x,y,y'\right)\,dx=\ell\in\mathbb{R}.$$

(6) Euler-Lagrange equation

$$(F-\lambda(x)g)_y-\frac{d}{dx}(F-\lambda(x)g)_{y'}=0,$$
$$(F-\lambda(x)g)_z-\frac{d}{dx}(F-\lambda(x)g)_{z'}=0$$

with Lagrange multiplier $\lambda(x)$ (which is a real-valued function of x) for the functional

$$J[y,z]=\int_{x=x_1}^{x_2} F\left(x,y,z,y',z'\right)\,dx$$

subject to the pointwise constraint

$$g\left(x,y,z\right)=0.$$

(7) The canonical variables x, y, p and the Hamiltonian H for the functional

$$J[y] = \int_{x=x_1}^{x_2} F(x, y, y')\, dx$$

are given by $p = F_{y'}$ and $H = -F + y'p$. The canonical differential equations are

$$\frac{dy}{dx} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dx} = -\frac{\partial H}{\partial y}.$$

(8) The action integral for the functional

$$J[y] = \int_{x=x_1}^{x_2} F(x, y, y')\, dx$$

is the integral

$$\int_{x_1}^{x_2} F(x, y, y')\, dx$$

computed along an extremal.

(9) The canonical transformation defined by the generating function $\Phi(x, y, Y)$ is given by

$$p = \frac{\partial \Phi}{\partial y}, \quad P = -\frac{\partial \Phi}{\partial Y}, \quad H^* = H + \frac{\partial \Phi}{\partial x}.$$

The canonical transformation defined by the generating function $\Psi(x, y, P)$ is given by

$$p = \frac{\partial \Psi}{\partial y}, \quad Y = \frac{\partial \Psi}{\partial P}, \quad H^* = H + \frac{\partial \Psi}{\partial x}.$$

(10) Legendre's necessary condition for a local minimum for the variational problem of the functional

$$J[y] = \int_{x=x_1}^{x_2} F(x, y, y')\, dx$$

is $P(x) \geq 0$ and the Jacobi differential equation is

$$-\frac{d}{dx}(Ph') + Qh = 0$$

with unknown function $h = h(x)$, where

$$P(x) = \frac{1}{2} F_{y'y'}, \qquad Q(x) = \frac{1}{2}\left(F_{yy} - \left(\frac{d}{dx} F_{yy'}\right)\right).$$