Solutions to Problems in
Math 115 First Mid-Term Examination
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Science Center 411

Answers have to be expressed in terms of real-valued functions of real variables, except that in Problem 4 there is no need to explicitly expand the real part and the imaginary part of a rational function. Precise meanings of multi-valued inverse functions have to be specified.

Some formulas are provided on the back of this page. Not all of them are necessary for the test.

SOME FORMULAS

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

$$\cot(x+iy) = \frac{\sin 2x}{2\left(\sin^2 x + \sinh^2 y\right)} - i\,\frac{\sinh 2y}{2\left(\sin^2 x + \sinh^2 y\right)}.$$

The equation for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is equivalent to

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a,$$

where $c = \sqrt{a^2 + b^2}$.

Problem 1. Use the theory of residues to compute

$$\int_{x=-\infty}^{\infty} \frac{\cos x\, dx}{\left(x^2+4\right)^2}.$$

Solution. The degree of the polynomial in the denominator is 4 and that of the polynomial in the numerator is 0 and so the former is at least two more than the latter. Moreover, the denominator $\left(x^2+4\right)^2$ has no zero on the real line. We can use integration of

$$\frac{e^{iz}}{\left(z^2+4\right)^2}$$

over the contour of the boundary of the upper half disk of radius R centered at the origin as $R \to \infty$ to get

$$\int_{x=-\infty}^{\infty} \frac{\cos x\, dx}{\left(x^2+4\right)^2} = \operatorname{Re}\left(2\pi i \sum_{\operatorname{Im}\zeta>0} \operatorname{Res}_{\zeta} \frac{e^{iz}}{\left(z^2+4\right)^2}\right).$$

There is only one point where we need to compute the residue. The point is $z = 2i$ and the pole order there is 2.

$$\begin{aligned}\operatorname{Res}_{z=2i} \frac{e^{iz}}{\left(z^2+4\right)^2} &= \frac{d}{dz}\left(\left(z-2i\right)^2 \frac{e^{iz}}{\left(z^2+4\right)^2}\right)\Bigg|_{z=2i} = \frac{d}{dz}\left(\frac{e^{iz}}{\left(z+2i\right)^2}\right)\Bigg|_{z=2i} \\ &= \left(\frac{ie^{iz}}{\left(z+2i\right)^2} - 2\frac{e^{iz}}{\left(z+2i\right)^3}\right)\Bigg|_{z=i} = \frac{ie^{-2}}{\left(4i\right)^2} - 2\frac{e^{-2}}{\left(4i\right)^3} = -\frac{3i}{32e^2}.\end{aligned}$$

Our final answer is

$$\int_{x=-\infty}^{\infty} \frac{\cos x\, dx}{\left(x^2+4\right)^2} = \operatorname{Re}\left(2\pi i\left(-\frac{3i}{32e^2}\right)\right) = \frac{3\pi}{16e^2}.$$

Problem 2. Let $0 < \alpha < 1$. Use the theory of residues and the branch of a holomorphic function on $\mathbb{C}$ minus the interval $[0,1]$ to compute

$$\int_{x=0}^{1} \frac{dx}{x^{\alpha}\left(1-x\right)^{1-\alpha}\left(x^2+1\right)}.$$

Solution. We introduce the following function

$$g(z) = z^{-\alpha}(1-z)^{\alpha-1}.$$

We would like to define an appropriate branch for this function.

How to Define a Branch. This was already given in the lecture notes. We are reproduce its definition here to remind ourselves how the branch is defined and how the numerical values for the angles of the relevant complex variables are chosen.

First we choose a branch for the function $z^{-\alpha}$ and then choose a branch for the function $(1-z)^{1-\alpha}$ and then put the two branches together.

To define a branch for $z^{-\alpha}$, we take away the slit $[0, \infty) \subset \mathbb{R}$ so that we restrict the numerical value for the angle θ in the polar representation of $z = re^{i\theta}$ to $0 < \theta < 2\pi$ and for such a restriction of the value of θ the value of $z^{-\alpha}$ is defined to be $r^{-\alpha}e^{-i\alpha\theta}$.

To define a branch for $(1-z)^{\alpha-1}$, we take away the slit $(-\infty, 0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ so that we restrict the numerical value for the angle φ in the polar representation of $1-z = \rho e^{i\varphi}$ to $-\pi < \varphi < \pi$ and for such a restriction of the value of φ the value of $(1-z)^{\alpha-1}$ is defined to be $\rho^{\alpha-1}e^{i(\alpha-1)\varphi}$. Note that taking away the slit $(-\infty, 0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ is the same as taking away $[1, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable z, because a translation of adding -1 to the variable $1-z$ moves the slit $(-\infty, 0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ to the slit $(-\infty, -1] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $-z$ and the transformation $z \mapsto -z$ moves the slit $(-\infty, -1] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $-z$ to the slit $[1, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable z. Note that the definition of $(1-z)^{\alpha-1}$ uses only the numerical value φ of the angle in the polar representation $1-z = \rho e^{i\varphi}$ and we do not have to choose a numerical value for he angle in the polar representation of z.

When we take the product of the branch $z^{-\alpha}$ and the branch $(1-z)^{\alpha-1}$, the slit $[0, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the variable z has to be excluded. However, we can put back the slit $(1, \infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the variable z into the domain of definition of the product of the branch $z^{-\alpha}$ and the branch $(1-z)^{\alpha-1}$ for the following reason. When z is just above the slit $(1, \infty) \subset \mathbb{R}$, the value of θ is 0 and the value of φ is $-\pi$ (corresponding to the numerical value of the angle of $1-z$ being $-\pi$) and as a consequence the value of $g(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}$. Now we consider the situation when z is just below the slit $(1, \infty) \subset \mathbb{R}$. When z is just below the slit $(1, \infty) \subset \mathbb{R}$, the value of θ is 2π and the value of φ is π (corresponding to the numerical value of the angle of $1-z$ being π) and as a consequence the value of $g(z) = z^{-\alpha}(1-z)^{\alpha-1}$

is

$$
\begin{aligned}
&x^{-\alpha}e^{-i2\alpha\pi}(x-1)^{\alpha-1}e^{i(\alpha-1)\pi}\\
&= x^{-\alpha}(x-1)^{\alpha-1}e^{-i\alpha\pi-i\pi}\\
&= x^{-\alpha}(x-1)^{\alpha-1}e^{-i\alpha\pi+i\pi}
\end{aligned}
$$

which again is equal to $x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}$. It means that the function $g(z) = z^{-\alpha}(1-z)^{\alpha-1}$ which is holomorphic on $\mathbb{C} - [0, \infty)$ can be extended to be a continuous function on $\mathbb{C} - [0, 1]$.

We now use the following statement: A continuous function on a domain which is holomorphic outside a line-segment in the domain must be holomorphic on the entire domain. From this statement it follows that the function $g(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is holomorphic on $\mathbb{C} - [0, 1]$.

Computation After Choice of Branch. Now that we have finished reproducing here the definition for the chosen branch from the lecture notes, we go back to the computation of our definite integral

$$\int_{x=0}^{1} \frac{dx}{x^{\alpha}(1-x)^{1-\alpha}\,(x^2+1)}.$$

Let C_R be the circle of radius R centered at the origin in the counterclockwise sense and let Γ_r be composed of the following four pieces: the right-half of the circle $|z-1| = r$ in the counterclockwise sense, the line-segment joining $1 + ri$ to ri, the left-half of the circle $|z| = r$ in the counterclockwise sense, and the line-segment joining $-ri$ to $1 - ri$.

We introduce the function

$$f(z) = \frac{g(z)}{z^2+1}$$

and apply Cauchy's theorem to the holomorphic function $f(z)$ on the domain enclosed by C_R and Γ_r for $R > 0$ sufficiently large and for $r > 0$ sufficiently small. Then

$$\int_{C_R} f(z)dz = \int_{\Gamma_r} f(z)dz + 2\pi i \left(\mathrm{Res}_{z=i}\, f(z) + \mathrm{Res}_{z=-i}\, f(z)\right).$$

The integral

$$\int_{C_R} f(z)dz$$

approaches 0 as $R \to \infty$, because

$$\sup_{z \in C_R} |f(z)| = O\left(\frac{1}{R^3}\right)$$

and the length of C_R is $O(R)$ as $R \to \infty$. In order to compute the limit of

$$\int_{\Gamma_r} f(z) dz$$

as $r \longrightarrow 0$, we determine the value of $g(x)$ for x just above $(0,1)$ and the value of $g(x)$ for x just below $(0,1)$.

Use Branch Values to Compute Contour Integral Around the Cut. To compute the value of $g(x)$ for x just above $(0,1)$, we observe that for x just above $(0,1)$ the value of θ is 0 and the value of φ is 0 (corresponding to the numerical value of the angle of $1-z$ being 0) and as a consequence the value of $g(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}(x-1)^{\alpha-1}$. Likewise, to compute the value of $g(x)$ for x just below $(0,1)$, we observe that for x just below $(0,1)$ the value of θ is 2π and the value of φ is 0 (corresponding to the numerical value of the angle of $1-z$ being 0) and as a consequence the value of $g(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}e^{-i\alpha 2\pi}(x-1)^{\alpha-1}$. Thus

$$\lim_{r \to 0} \int_{\Gamma_r} f(z) dz$$

is equal to

$$\begin{aligned} e^{-i\alpha 2\pi} \int_{x=0}^{1} \frac{dx}{x^\alpha (1-x)^{1-\alpha} (x^2+1)} &- \int_{x=0}^{1} \frac{dx}{x^\alpha (1-x)^{1-\alpha} (x^2+1)} \\ &= \left(e^{-i\alpha 2\pi} - 1\right) \int_{x=0}^{1} \frac{dx}{x^\alpha (1-x)^{1-\alpha} (x^2+1)}. \end{aligned}$$

Residue Computation. We now compute the two residues.

$$\mathrm{Res}_{z=i} f(z) = \lim_{z \to i} \frac{g(z)}{(z+i)} = -\frac{i}{2} g(i).$$

$$\mathrm{Res}_{z=-i} f(z) = \lim_{z \to -i} \frac{g(z)}{(z-i)} = \frac{i}{2} g(-i).$$

Thus we end up with

$$\int_{x=0}^{1} \frac{dx}{x^{\alpha}(1-x)^{1-\alpha}\left(x^{2}+1\right)}=\frac{\pi}{1-e^{-i\alpha 2\pi}}\left(g(i)-g(-i)\right).$$

To compute the value of $g(i)$, we observe that the value of θ is $\frac{\pi}{2}$ and the value of φ is $-\frac{\pi}{4}$ (corresponding to the numerical value of the angle of $1-z$ being $-\frac{\pi}{4}$) and as a consequence the value of $g(z)=z^{-\alpha}(1-z)^{\alpha-1}$ at $z=i$ is

$$e^{-i\frac{\pi}{2}\alpha}2^{\frac{\alpha-1}{2}}e^{-i\frac{\pi}{4}(\alpha-1)}=2^{\frac{\alpha-1}{2}}e^{-i\frac{3\pi}{4}\alpha+i\frac{\pi}{4}}.$$

Likewise, to compute the value of $g(-i)$, we observe that the value of θ is $\frac{3\pi}{2}$ and the value of φ is $\frac{\pi}{4}$ (corresponding to the numerical value of the angle of $1-z$ being $\frac{\pi}{4}$) and as a consequence the value of $g(z)=z^{-\alpha}(1-z)^{\alpha-1}$ at $z=-i$ is

$$e^{-i\alpha\frac{3\pi}{2}}2^{\frac{\alpha-1}{2}}e^{i\frac{\pi}{4}(\alpha-1)}=2^{\frac{\alpha-1}{2}}e^{-i\frac{5\pi}{4}\alpha-i\frac{\pi}{4}}.$$

Final Answer. As a result, we have

$$\begin{aligned}
&\int_{x=0}^{1} \frac{dx}{x^{\alpha}(1-x)^{1-\alpha}\left(x^{2}+1\right)} \\
&=\frac{2^{\frac{\alpha-1}{2}}\pi}{1-e^{-i\alpha 2\pi}}\left(e^{-i\frac{3\pi}{4}\alpha+i\frac{\pi}{4}}-e^{-i\frac{5\pi}{4}\alpha-i\frac{\pi}{4}}\right) \\
&=\frac{2^{\frac{\alpha-1}{2}}\pi}{e^{i\alpha\pi}-e^{-i\alpha\pi}}\left(e^{i\frac{\pi}{4}\alpha+i\frac{\pi}{4}}-e^{-i\frac{\pi}{4}\alpha-i\frac{\pi}{4}}\right) \\
&=\frac{2^{\frac{\alpha-1}{2}}\sin\left(\frac{\pi}{4}(\alpha+1)\right)}{\sin(\alpha\pi)}.
\end{aligned}$$

Problem 3. Use the theory of residues and the cotangent function to compute

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^{2}+n+1}.$$

Solution. We recall the following general procedure given in the lecture notes. Suppose $f(z)$ is a rational function whose poles are simple nonintegers $a_1,\cdots,a_k$ with residues $b_1,\cdots,b_k$ such that the degree of the denominator of f is at least two more than that of its numerator. Let C_n be the square with corners at

$$\left(n+\frac{1}{2}\right)(\pm 1\pm i).$$

The integral

$$\int_{C_n} \pi \, \cot \pi z \, f(z) dz$$

goes to zero as $n \to \infty$. Hence

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{\nu=1}^{k} b_\nu \cot \, \pi \, a_\nu.$$

In our case the holomorphic function is

$$f(z) = \frac{1}{z^2+z+1} = \frac{1}{\left(z - e^{i\frac{2\pi}{3}}\right)\left(z - e^{-i\frac{2\pi}{3}}\right)}$$

whose two poles a_1, a_2 at the non-real cubic roots of unity

$$e^{i\frac{2\pi}{3}} = \frac{-1+i\sqrt{3}}{2} \qquad \text{and} \qquad e^{-i\frac{2\pi}{3}} = \frac{-1-i\sqrt{3}}{2}$$

are both simple. The residue b_1 at $a_1 = e^{i\frac{2\pi}{3}}$ is

$$\frac{1}{e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}} = \frac{1}{i\sqrt{3}}$$

and the residue b_2 at $a_2 = e^{-i\frac{2\pi}{3}}$ is

$$\frac{1}{e^{-i\frac{2\pi}{3}} - e^{i\frac{2\pi}{3}}} = \frac{-1}{i\sqrt{3}}.$$

We have to compute the value of the cotangent function at the two poles $e^{i\frac{2\pi}{3}}$ and $e^{-i\frac{2\pi}{3}}$. After we convert the polar representation of the two poles by Cartesian representations, we use the formula for the value of the cotangent function at a complex number whose derivation is given below. Since

$$\cot \, \pi \, e^{i\frac{2\pi}{3}} = \cot \left(\pi \frac{-1+i\sqrt{3}}{2}\right),$$

we can get its real and imaginary parts from the given formula

$$(\natural) \qquad \cot\,(x+iy) = \frac{\sin 2x}{2\left(\sin^2 x + \sinh^2 y\right)} - i\,\frac{\sinh 2y}{2\left(\sin^2 x + \sinh^2 y\right)}.$$

As we will see later, only the imaginary part is needed.

$$\operatorname{Im}\left(\cot\,\pi\, e^{i\frac{2\pi}{3}}\right)=\operatorname{Im}\left(\cot\left(\pi\frac{-1+i\sqrt{3}}{2}\right)\right)=-\frac{\sinh\sqrt{3}\pi}{2\left(\sin^2\frac{\pi}{2}+\sinh^2\frac{\sqrt{3}\pi}{2}\right)}.$$

The final answer $-\pi\left(b_1\cot\pi a_1+b_2\cot\pi a_2\right)$ is now

$$\begin{aligned}
&-\frac{\pi}{e^{i\frac{2\pi}{3}}-e^{-i\frac{2\pi}{3}}}\left(\cot\,\pi\, e^{i\frac{2\pi}{3}}-\cot\,\pi\, e^{-i\frac{2\pi}{3}}\right)\\
&=-\frac{\pi}{e^{i\frac{2\pi}{3}}-e^{-i\frac{2\pi}{3}}}\left(2i\operatorname{Im}\left(\cot\,\pi\, e^{i\frac{2\pi}{3}}\right)\right)\\
&=-\frac{\pi}{e^{i\frac{2\pi}{3}}-e^{-i\frac{2\pi}{3}}}\left(-2i\,\frac{\sinh\sqrt{3}\pi}{2\left(\sin^2\frac{\pi}{2}+\sinh^2\frac{\sqrt{3}\pi}{2}\right)}\right)\\
&=\frac{\pi\sinh\sqrt{3}\pi}{\sqrt{3}\left(1+\sinh^2\frac{\sqrt{3}\pi}{2}\right)}.
\end{aligned}$$

Derivation of Formula for Value of Cotangent Function at a Complex Number. The formula (♮) is provided with the test problems. We would like to give also here the details of its derivation. We start out with the values of the sine and cosine functions at a complex number. First when the complex number is purely imaginary.

$$\begin{aligned}
\cos iy&=\frac{e^{i(iy)}+e^{-i(iy)}}{2}=\frac{e^{-y}+e^{y}}{2}=\cosh y.\\
\sin iy&=\frac{e^{i(iy)}-e^{-i(iy)}}{2i}=\frac{e^{-y}-e^{y}}{2i}=i\sinh y\\
&.
\end{aligned}$$

Then we use the addition formulas to get the case of a general complex number.

$$\begin{aligned}
\sin\left(x+iy\right)&=\sin x\cos iy+\cos x\sin iy=\sin x\cosh y+i\cos x\sinh y.\\
\cos\left(x+iy\right)&=\cos x\cos iy-\sin x\sin iy=\cos x\cosh y-i\sin x\sinh y.
\end{aligned}$$

We put them together into a formula for the cotangent function.

$$\cot\left(x+iy\right)=\frac{\cos\left(x+iy\right)}{\sin\left(x+iy\right)}=\frac{\cos x\cosh y-i\sin x\sinh y}{\sin x\cosh y+i\cos x\sinh y}$$

$$= \frac{(\cos x \cosh y - i \sin x \sinh y)(\sin x \cosh y - i \cos x \sinh y)}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \frac{\left(\sin x \cos x \cosh^2 y - \sin x \cos x \sinh^2 y\right) - i\left(\sin^2 x \sinh y \cosh y + \cos^2 x \sinh y \cosh y\right)}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \frac{\sin x \cos x - i \sinh y \cosh y}{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$

$$= \frac{\sin x \cos x - i \sinh y \cosh y}{\sin^2 x \left(1 + \sinh^2 y\right) + \cos^2 x \sinh^2 y}$$

$$= \frac{\sin x \cos x - i \sinh y \cosh y}{\sin^2 x + \sinh^2 y}$$

Finally we separate the real and imaginary parts and use the double angles formula for both the sine function and the hyperbolic sine function.

$$\operatorname{Re}(\cot(x+iy)) = \frac{\sin x \cos x}{\sin^2 x + \sinh^2 y} = \frac{\sin 2x}{2\left(\sin^2 x + \sinh^2 y\right)}.$$

$$\operatorname{Im}(\cot(x+iy)) = -\frac{\sinh y \cosh y}{\sin^2 x + \sinh^2 y} = -\frac{\sinh 2y}{2\left(\sin^2 x + \sinh^2 y\right)}.$$

Problem 4. On the open upper half-disk

$$D := \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1,\ \operatorname{Im} z > 0 \right\}$$

compute the steady temperature distribution T as a bounded harmonic function on D such that

(i) T has constant boundary value 1 on $(-1, 0)$,

(ii) the interval $(0, 1)$ as part of the boundary of D is insulated in the sense that the normal derivative of T is identically zero at every point of $(0, 1)$.

(iii) T has constant boundary value -1 on

$$C := \left\{ z \in \mathbb{C} \,\middle|\, |z| = 1,\ \operatorname{Im} z > 0 \right\}.$$

Solution. Use $z_1 = \frac{z-1}{z+1}$ which maps D to the third quadrant with $T = 1$ on $(-\infty, -1)$ and the interval $(-1, 0)$ is insulated and $T = -1$ the upper imaginary axis. The transformation $z_2 = z_1^2$ sends the third quadrant to the lower half-plane with $T = -1$ on $(1, \infty)$ and the interval $(0, 1)$ is insulated and $T = 1$ on the negative real axis $(-\infty, 0)$. The transformation $z_3 = -2z_2 + 1$ sends the lower half-plane to the upper half-plane with $(-1, 1)$ insulated and $T = 1$ on $(1, \infty)$ and $T = -1$ on $(-\infty, -1)$. Finally the inverse sine function $z_4 = \sin^{-1} z_3$ maps the upper half-plane to the upper half strip with base $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The choice of the branch $\sin^{-1} z_3$ is that its real part lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The temperature is given by $T = \frac{2}{\pi}\mathrm{Re}\, z_4$. Recall that

$$\mathrm{Re}\left(\sin^{-1}(x+iy)\right) = \sin^{-1}\frac{1}{2}\left(\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}\right).$$

Hence

$$\begin{aligned}\mathrm{Re}\, z_4 &= \mathrm{Re}\, \sin^{-1} z_3 = \mathrm{Re}\, \sin^{-1}(-2z_2+1) \\ &= \mathrm{Re}\, \sin^{-1}\left(-2{z_1}^2+1\right) = \mathrm{Re}\, \sin^{-1}\left(-2\left(\frac{z-1}{z+1}\right)^2+1\right) \\ &= \mathrm{Re}\, \sin^{-1}\left(\frac{-2(z-1)^2+(z+1)^2}{(z+1)^2}\right) \\ &= \mathrm{Re}\, \sin^{-1}\left(\frac{-z^2+6z-1}{(z+1)^2}\right).\end{aligned}$$

The final answer in the form

$$T = \frac{2}{\pi}\sin^{-1}\left(\frac{1}{2}\left(\sqrt{(x_3+1)^2+{y_3}^2} - \sqrt{(x_3-1)^2+{y_3}^2}\right)\right)$$

with the range of $\sin^{-1}$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, where

$$z_3 = x_3 + iy_3 = \frac{-z^2+6z-1}{(z+1)^2}$$

with $z = x + iy$, is considered complete.

The final step uses the alternative description of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

by

$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$$

with $c = \sqrt{a^2 + b^2}$ as follows. From

$$\begin{aligned} x_3 + iy_3 &= \sin(x_4 + iy_x) \\ &= \sin x_4 \cos(iy_4) + \cos x_4 \sin(iy_4) \\ &= \sin x_4 \cosh y_4 + i \cos x_4 \sinh y_4 \end{aligned}$$

it follows that

$$x_3 = \sin x_4 \cosh y_4 \qquad \text{and} \qquad y_3 = \cos x_4 \sinh y_4$$

and, in order to solve for x_4 in terms of x_2 and y_3, we use

$$\cosh^2 y_4 - \sinh^2 y_4 = 1$$

to eliminate the variable y_4 from the above two equations and get

$$(\sharp) \qquad \frac{{x_3}^2}{\sin^2 x_4} - \frac{{y_3}^2}{\cos^2 x_4} = 1.$$

By applying the above alternative description of the hyperbola with

$$a = |\sin x_4| \qquad \text{and} \qquad b = |\cos x_4|,$$

we get

$$c = \sqrt{a^2 + b^2} = \sqrt{\sin^2 x_4 + \cos^2 x_4} = 1$$

and the alternative description of ($\sharp$) is

$$\sqrt{(x_3 + 1)^2 + {y_3}^2} - \sqrt{(x_3 - 1)^2 + {y_3}^2} = \pm 2 |\sin x_4|$$

and as a result

$$x_4 = \sin^{-1}\left(\frac{1}{2}\left(\sqrt{(x_3 + 1)^2 + {y_3}^2} - \sqrt{(x_3 - 1)^2 + {y_3}^2}\right)\right).$$

Remark. It is also possible to map the third quadrant with variable z_1 directly to the vertical upper half-strip $\left\{ -\frac{\pi}{2} < x < 0,\ y > 0 \right\}$ of width $\frac{\pi}{2}$ by the inverse sine function, bypassing the steps of using z_2 and z_3.