Math 116: Section Notes

Michael McElroy

24 October 2003

Theorem. *If we are in a Hilbert space H and wish to find a point in M which minimize the distance from $x \in H$ to the closed subspace M, then $m \in M$ exists, is unique, and $x - m \perp M$.*

What does it mean for $x - m \perp M$? How can we use this to solve problems?

Problem 1. Luenberger 3.7. Let M and N be orthogonal closed subspaces of a Hilbert space H and let $x \in H$ be given. Show that $M \oplus N$ is closed and that the orthogonal projection of x onto $M \oplus N$ is equal to $m + n$, where m and n are the orthogonal projections of x onto M and N respectively.

Proof. Why do we care that $M \oplus N$ is closed? So that we may apply the projection theorem, and to know that a minimizer exists. We want limits in $M \oplus N$ to remain in the set. This follows from the closure of M and N.

Now, M and N are subspaces, so $0 \in M, N$. M and N are orthogonal, so for all $m' \in M$ and $n' \in N$ we have $(m'|n') = 0$. This is what $M \perp N$ means. We have that $(x - m|m - m') = 0$, but since $m - m' = m'' \in M$, we may as well write $(x - m|m) = (x - m|m') = 0$. Same for N, $(x - n|n) = (x - n|n') = 0$. Now, we want to show that $x - (m + n) \perp M \oplus N$, so that $(x - m - n|m + n - m' - n') = 0$. Expanding, we have

$$(x - m - n|m + n - m' - n') = (x - m|m - m') + (x - n|n - n') - (n|m - m') - (m|n - n').$$

Since m and n are minimizers to M and N, the first two terms equal zero, and since $M \perp N$, the last two terms equal zero. □