Math 116

Some Solutions for Assignment IV

3.13.1)

($\Rightarrow$) Suppose $|(x,y)| = \|x\|\|y\|$. Then

$$(x,x) = \left(\frac{(x,y)}{(y,y)}\right)(y,x) = \lambda(y,x)$$

for $\lambda = \frac{(x,y)}{(y,y)}$. But

$$\begin{aligned}(x-\lambda y, x-\lambda y) &= (x,x) - \lambda(y,x) - \overline{\lambda}(x,y) + |\lambda|^2(y,y) \\ &= 0 \\ &\Rightarrow x - \lambda y = 0.\end{aligned}$$

($\Leftarrow$) Suppose $\alpha x + \beta y = 0$, $\alpha \neq 0$. Then $x = \lambda y$ with $\lambda = -\frac{\beta}{\alpha}$. But then

$$\begin{aligned}|(x,y)| &= |(\lambda y, y)| \\ &= |\lambda||(y,y)| \\ &= |\lambda|\|y\|^2 \\ &= \|\lambda y\|\|y\| \\ &= \|x\|\|y\|.\end{aligned}$$

3.13.3) Clearly H is a vector space over the reals. We must check that the inner product is well-defined:

Symmetry

$$\begin{aligned}(A,B) &= \operatorname{tr}(A^TQB) \\ &= \operatorname{tr}(B^TQA) \\ &= (B,A).\end{aligned}$$

Linearity

$$\begin{aligned}(A+B,C) &= \operatorname{tr}((A+B)^TQC) \\ &= \operatorname{tr}((A^T+B^T)QC) \\ &= \operatorname{tr}(A^TQC) + \operatorname{tr}(B^TQC) \\ &= (A,C) + (B,C).\end{aligned}$$

Homogeneity

$$\begin{aligned}(\lambda A, B) &= \mathrm{tr}((\lambda A)^T QC) \\ &= \mathrm{tr}(\lambda A^T QC) \\ &= \lambda \mathrm{tr}(A^T QC) \\ &= \lambda (A, B).\end{aligned}$$

Positive Definiteness Since Q is a positive definite matrix, there is an orthogonal matrix P and a diagonal matrix D with positive entries along the diagonal s.t. $Q = P^T DP$. Thus

$$\begin{aligned}(A, A) &= \mathrm{tr}(A^T QA) \\ &= \mathrm{tr}(A^T P^T DPA) \\ &= \mathrm{tr}((PA)^T D(PA)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{m}(PA)_{ki} D_{kj} (PA)_{ji} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}(PA)_{ji}^2 D_{jj} \\ &\geq 0.\end{aligned}$$

Now, if A is the zero matrix, then clearly $(A, A) = 0$. Conversely, if $(A, A) = 0$, we must have PA is the zero matrix by positivity of D_{ii} for $1 \leq i \leq m$. But then $P^T PA = A$ is also the zero matrix.

Finally, H is complete because it is finite dimensional.

3.13.5) We use the Classical Projection theorem with $H = L_2[-1, 1]$ and M the space of polynomials with degree at most 1. Then we want $x(t) = a + bt$ s.t. $t^2 - x(t)$ is orthogonal to M. That is, for all $c, d \in \mathbb{R}$,

$$\int_{-1}^{1} (t^2 - a - bt)(c + dt)\ \mathrm{d}t = \frac{1}{3}\,(2c - 2bd - 6ac) = 0.$$

Substituting $c = 0$, we see that $b = 0$. Substituting $d = 0$, we see that $a = \frac{1}{3}$. So $x(t) = \frac{1}{3}$.

3.13.6)

a) Let $H = L_2[0, 1]$ be our Hilbert space, and define the set

$$M = \{x \in N \mid Ix = 0\}.$$

Here N denotes the space of polynomials with degree n or less, and I denotes the linear operator $Ix = \int_0^1 x(t)\ dt$. Then linearity of I makes

M a subspace of H, and finite dimensionality implies that M is closed. Thus we can use the Classical Projection theorem to tell us that there is a unique solution.

b) Take M, N, and H as above. Then N is closed because it has finite dimension. If $q \in N$ minimizes $\|x-q\|^2$, then $x-q$ is orthogonal to N by the Projection theorem. Thus $(x,n)=(q,n)$ for all $n \in N$. Similarly, if $p \in M$ minimizes $\|q-p\|^2$, then $q-p$ is orthogonal to M, and $(q,m)=(p,m)$ for all $m \in M$. Now, take any $m \in M$. Then $m \in N$ also, and we have

$$(x-p,m)=(x,m)-(p,m)=(q,m)-(q,m)=0.$$

Since $m \in M$ was abitrary, we have that $x-p$ is orthogonal to M. By the Projection theorem, p is the unique solution to the minimization problem.

3.13.7) First, we show $M \oplus N$ is closed. Assume the sequence $x_k \to x$, with $x_k \in M \oplus N$ for each k and $x \in H$. Then we can write $x_k = m_k + n_k$ for each k, with $m_k \in M$ and $n_k \in N$. We also write $x = m + n$, with $m \in M$ and $n \in M^\perp$. Then the Pythagorean theorem says

$$\|x-x_k\| = \|(m-m_k)+(n-n_k)\| = \|m-m_k\| + \|n-n_k\|,$$

since $M \subset N^\perp$ and $N \subset M^\perp$. Thus $m_k \to m \in M$, since M is closed, and $n_k \to n \in N$, since N is closed. So $x = m+n \in M \oplus N$, as desired. The Classical Projection theorem says that there is a unique orthogonal projection of x onto $M \oplus N$, M, and N. Assume the projection of x onto M is m_0, and onto N is n_0. Then let $m+n \in M \oplus N$. We have:

$$(x-(m_0+n_0), m+n) = (x-m_0,m)-(n_0,m)+(x-n_0,n)-(m_0,n)=0$$

since each of the four terms is 0. The Projection theorem then says that m_0+n_0 is the unique orthogonal projection of x onto $M \oplus N$.

3.13.12) Let N denote the space of polynomials of degree $n-1$ or less. Then N is a closed subspace of the Hilbert space $L_2[0,T]$, so the Projection theorem says that there exists a unique $p(T,t) \in N$ s.t. $x(t)-p(T,t)$ is orthogonal to N. So taking $\{t^{i-1}\}_{i=1}^n$ as our basis for N, we have, for $1 \le i \le n$,

$$\int_0^T (x(t)-p(T,t))\, t^{i-1}\, dt = \int_0^T \left(x(t) - \sum_{j=1}^n a_j(T) t^{j-1} \right) t^{i-1}\, dt = 0.$$

Rearranging, for each $1 \le i \le n$, we have

$$\int_0^T t^{i-1} x(t)\, dt = \sum_{j=1}^n a_j(T) \int_0^T t^{i+j-2}\, dt.$$

Now differentiate both sides by T, using the Liebniz rule and the fundamental theorem of calculus to get:

$$\begin{aligned} T^{i-1}x(T) &= \left[\sum_{j=1}^{n} a_j'(T)\frac{T^{j+i-1}}{i+j-1}\right] + T^{i-1}\left[\sum_{j=1}^{n} a_j(T)T^{j-1}\right] \\ &= T^{i-1}\left[\sum_{j=1}^{n} a_j'(T)\frac{T^{j}}{i+j-1}\right] + T^{i-1}p(T,T). \end{aligned}$$

Dividing by T^{i-1} and rearranging, we have, for $1 \leq i \leq n$:

$$x(T) - p(T,T) = \varepsilon(T) = \sum_{j=1}^{n} a_j'(T)\frac{T^j}{i+j-1}.$$

Now if we call A the $n\times n$ matrix s.t. $A_{ij} = \frac{T^j}{i+j-1}$, b the $n \times 1$ vector with $b_j = a_j'(T)$, and c the $n\times 1$ vector with $c_j = \varepsilon(T)$, then we have $Ab = c$. Cramer's rule give us $a_j'(T) = \frac{|\hat{A}_j|}{|A|}$, where $\hat{A}_j$ is the matrix A with the jth column replaced by c. Now call $\hat{B}^{ij}$ the $(n-1)\times(n-1)$ matrix formed by deleting the ith row and jth column of A. Then

$$\begin{aligned} a_j'(T) &= \frac{\sum_{i=1}^{n} \varepsilon(T)(-1)^i|\hat{B}^{ij}|}{\sum_{i=1}^{n}\frac{T^j}{i+j-1}(-1)^i|\hat{B}^{ij}|} \\ &= \frac{\varepsilon(T)}{T^j}\left[\frac{\sum_{i=1}^{n}(-1)^i|\hat{B}^{ij}|}{\sum_{i=1}^{n}\frac{(-1)^i}{i+j-1}|\hat{B}^{ij}|}\right]. \end{aligned}$$

If we call $b_j = \frac{\sum_{i=1}^{n}(-1)^i|\hat{B}^{ij}|}{\sum_{i=1}^{n}\frac{(-1)^i}{i+j-1}|\hat{B}^{ij}|}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}T}a_j(T) = \frac{b_j\varepsilon(T)}{T^j}.$$

3.13.16)

(⇒) Assume that the e_i are complete. Then $x = \sum_{i=1}^{\infty}(x,e_i)e_i$, and $y = \sum_{i=1}^{\infty}(y,e_i)e_i$. We have:

$$\begin{aligned} (x,y) &= \left(\sum_{i=1}^{\infty}(x,e_i)e_i, \sum_{j=1}^{\infty}(y,e_j)e_j\right) \\ &= \sum_{i,j=1}^{\infty}(x,e_i)(e_j,y)(e_i,e_j) \\ &= \sum_{i=1}^{\infty}(x,e_i)(e_i,y). \end{aligned}$$

(⇐) Assume that $(x, y) = \sum_{i=1}^{\infty}(x, e_i)(e_i, y)$ for all $x,\ y \in H$. Then if there is a $z \in H$ s.t. $(z, e_i) = 0$ for all e_i, we have $(z, z) = \sum_{i=1}^{\infty}(z, e_i)(e_i, z) = 0$, so z must be the zero vector. So the e_i are complete.

Legendre Polynomials Consider the Hilbert space $L_2[-1, 1]$. The independent functions $1, t, t^2, \ldots$ generate the subspace of polynomials. Also, the closure of their span is equal to H by the Weierstrass Approximation thereom. We explicitly compute the first few orthonormal polynomials from the t^i using the Gram-Schmidt procedure. The first polynomial is the constant polynomial 1, so we have $f_0 = 1$. We normalize this by taking

$$e_0 = \frac{f_0}{\sqrt{(f_0, f_0)}} = \frac{1}{\sqrt{\int_{-1}^{1} \mathrm{d}t}} = \frac{1}{\sqrt{2}} = \sqrt{\frac{1}{2}} P_0,$$

where $P_0 = 1$, the first Legendre polynomial. Next, we take t, and construct

$$f_1 = t - (t, e_0)\, e_0 = t - \left(\int_{-1}^{1} \frac{t}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} = t.$$

We normalize this by taking

$$e_1 = \frac{f_1}{\sqrt{(f_1, f_1)}} = \frac{t}{\sqrt{\int_{-1}^{1} t^2\, \mathrm{d}t}} = \sqrt{\frac{3}{2}}\, t = \sqrt{\frac{3}{2}} P_1,$$

where $P_1 = t$, the second Legendre polynomial. Next we take t^2, and the Gram Schmidt procedure gives

$$f_2 = t^2 - \left(t^2, e_1\right) e_1 - \left(t^2, e_0\right) e_0 = t^2 - \frac{1}{3}.$$

We normalize again:

$$e_2 = \frac{f_2}{\sqrt{(f_2, f_2)}} = \frac{3}{2}\sqrt{\frac{5}{2}} \left(t^2 - \frac{1}{3}\right) = \sqrt{\frac{5}{2}} P_2.$$

Once more, the procedure yields:

$$f_3 = t^3 - \left(t^3, e_2\right) e_2 - \left(t^3, e_1\right) e_1 - \left(t^3, e_0\right) e_0 = t^3 - \frac{3}{5} t.$$

Normalizing this then gives:

$$e_3 = \frac{f_3}{\sqrt{(f_3, f_3)}} = \frac{5}{2}\sqrt{\frac{7}{2}} \left(t^3 - \frac{3}{5} t\right) = \sqrt{\frac{7}{2}} P_3.$$

Were we to continue this procedure, we could get a complete orthonormal set for the Hilbert space $L_2[-1, 1]$.