HOMEWORK #6 SOLUTIONS

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1. OVSM

5.14.4 (Sam Gansfried) Define $f : NVB[a,b] \to \mathbb{R}$ with $f(v) = v((a+b)/2)$. This is clearly linear, and

$$\|f(v)\| \le TV(v)$$

so it is bounded. If $C[a,b]$ were reflexive we could find x such that $\int_a^b x(t)dv(t) = v((a+b)/2)$ for all v. However, if we let v_α be a step function which is 0 before α and 1 after, we get the integral to be $x(\alpha)$. If $\alpha < (a+b)/2$ we have $v_\alpha((a+b)/2) = 1$, otherwise we have $v_\alpha((a+b)/2) = 0$, so we have that x is a step function that goes from 1 to 0 at $(a+b)/2$. But if we now let

$$v(t) = \left(t - \frac{3a+b}{4}\right)^2 - \frac{(b-a)^2}{16}$$

we get $\int_a^b x(t)dv(t)$ is the variation of v on $[a,(a+b)/2]$ (which is nonzero) while $v((a+b)/2) = 0$. So there is no such $x \in C[a,b]$ and therefore $C[a,b]$ is not reflexive.

5.14.6 We know that $d^2x/dt^2 = u(t)$, so integrating this twice we get the following equations:

$$\frac{dx}{dt}(T) = 1 + \int_0^T u(t)\,dt \qquad x(T) = x(0) + T + \int_0^T \int_0^t u(s)\,ds\,dt.$$

Switching the order of integration in the second integral and expanding we get that

$$x(T) = T\frac{dx}{dt}(T) - \int_0^T tu(t)\,dt.$$

In the optimal solution for x we will stop once we reach the origin, so $\frac{dx}{dt}(T) = 0$. Thus we now have the following three conditions:

$$x(T) = -\int_0^T tu(t)\,dt \qquad \int_0^T tu(t)\,dt = 0 \qquad |u(t)| \le 1.$$

For any fixed time T, let S_T be the set of functions u that satisfy the first two of these conditions. We want to find the minimum T' such that $S_{T'}$ contains a function u with $\max_t |u(t)| \le 1$. Thus we want to minimize T such that

$$\min_{u \in S_T} \max_t |u(t)| = 1.$$

Notice that the function $\min_{S_T} \max_t |u(t)|$ shrinks as a function of T; thus if we find some T such that the above holds, we will have found the global minimum.

However, we can reduce this to a problem in L_1*: we want to find the u in S_T such that $\langle 1, u\rangle = -1$ and $\langle t, u\rangle = 0$. But we know that then in L_∞,

$$\min_T \|u\|_\infty = \max_{\|a_1+a_2t\|_1\le 1} -a_1$$

with the optimal u aligned with the optimal $a_1 + a_2t$. Two functions in L_∞ and L_1 will be aligned if and only if $y(t) = r\mathrm{sgn}(x(t))$ for $r \in \mathbb{R}$. Thus we want a solution where we decelerate as fast as possible, and then break at the last minute.

Knowing this, we can solve the two constraints that we have to find that $T = 1+\sqrt{2}$.

5.14.10 First we will show that $M^* = X^*/M^\perp$. Consider a map in M^*, m^*. By Hahn-Banach there exists a map $\widetilde{m}^*$ such that $\widetilde{m}^*|M = m^*$ and such that $\|\widetilde{m}^*\| = \|m^*\|$. This map is well-defined on $X^*/M^\perp$, since if x^* is another such extension then $\widetilde{m}^* - x^*$ is zero on M, and therefore in $M^\perp$. It is also clearly linear. If we define $f : M^* \to X^*/M^\perp$ by $m^* \mapsto \widetilde{m}^*$, we claim that we have an isometric isomorphism.

Indeed, suppose that $\widetilde{m}^* = \widetilde{n}^*$. Then, on M, $\widetilde{m}^* - \widetilde{m}^* = 0$, which means that $m^* = n^*$, so the map is injective. Now consider $[x^*] \in X^*/M^\perp$. Then the functional $\widetilde{x^*|_M}$ will be the same on M as x^*, and so will be in $[x^*]$. Thus the map is surjective. So it only remains to show that it is an isometry. Notice that for any $x^* \in M^\perp$,

$$\|\widetilde{m}^* + x^*\| = \sup_{\|x\|=1} |(\widetilde{m}^* + x^*)(x)| \ge \sup_{\substack{\|x\|=1\\ x\in M}} |(\widetilde{m}^* + x^*)(x)| = \|m^*\|_M = \|\widetilde{m}^*\|.$$

Therefore, $\inf_{x^*\in M^\perp} \|\widetilde{m}^* + x^*\| = \|m^*\|$, so the map is an isometry.

Now we will show that $M^\perp = (X/M)^*$. We will show that the map defined byy $f(x^*)([y]) = x^*(y)$ is the isometric isomorphism we want. The map is clearly linear. It is well-defined because each x^* is 0 on M, so adding an element of M to y would not change the map. The map is injective because if $f(x^*) = 0$ then $x^* = 0$ on all elements of X, so $x^* = 0$. Consider a map $y^* \in (X/M)^*$. We can define x^* by $x^*(x) = y^*(x)$. This will be in $M^\perp$ since $[m] = [0]$ for all $m \in M$, and $f(x^*) = y^*$. So the map is surjective. Also,

$$\|f(x^*)\| = \sup_{\|[x]\|=1} |f(x^*)(x)| = \sup_{\|x\|=1} |x^*(x)| = \|x^*\|$$

so the map is an isometry.

5.14.11 We define $x^*(x) = \lim_{k\to\infty} x_k^*(x)$. We need to show that the right-hand side converges and that the left-hand side will be a linear functional. (It would clearly be bounded since the sequence of x_K^*'s is bounded.)

Let D be the dense set on which the x_k converge, and let M be the bound on $\{x_k^*\}$. Fix a point $x \in X$. We know that for any $\epsilon > 0$ there exists a point $y \in D$ such that $\|x - y\| < \epsilon/3M$ We know that there exists an N such that $|\langle y, x_n^*\rangle - \langle y, x_m^*\rangle| < \epsilon/3$ for all $m, n > N$. (Note that x^* is well-defined on D.) Then we know that

$$\begin{aligned} |\langle x, x_n^*\rangle - \langle x, x_m^*\rangle| &\le |\langle x, x_n^*\rangle - \langle y, x_n^*\rangle| + |\langle y, x_n^*\rangle - \langle y, x_m^*\rangle| + |\langle y, x_m^*\rangle - \langle x, x_m^*\rangle| \\ &\le \|x-y\| \cdot \|x_n^*\| + \epsilon/3 + \|y-x\| \cdot \|x_m\|^* \\ &< \epsilon. \end{aligned}$$

Thus we know that $\{\langle x, x_n^*\rangle\}$ is a Cauchy sequence in $\mathbb{R}$, so it converges, so x^* converges on all of X. Now we just need to show that it is linear. However, notice that

$$x^*(\alpha x+y) = \lim_{k\to\infty}(x_k^*(\alpha x+y)) = \lim_{k\to\infty}(\alpha x_k^*(x)+x_k^*(y)) = \alpha \lim_{k\to\infty} x_k^*(x)+\lim_{k\to\infty} x_k^*(y) = \alpha x^*(x)+x^*(y),$$

as desired. Notice that we can expand the limit because we know that each of the separate parts of the sum converges pointwise. Thus the sequence $\{x_k^*\}$ converges weak* to x^*.

5.14.15 Consider the linear map $F : X \to \mathbb{R}^n$ defined by $x \mapsto (f_1(x), \dots, f_n(x))$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$. We want to show that α is in the image of F if and only if for all $\lambda_1, \dots, \lambda_n$, if $\sum \lambda_i f_i = 0$, then $\sum \lambda_i \alpha_i = 0$.

The forward direction is easy. If there exists x such that $F(x) = \alpha$, then $\sum \lambda_i f_i = 0$ implies that $\sum \lambda_i f_i(x) = \sum \lambda_i \alpha_i = 0$.

Now suppose that we know that if $\sum \lambda_i f_i = 0$ then $\sum \lambda_i \alpha_i = 0$. We will show that α is in the image of F. Let $v_1, \dots, v_n$ be an orthonormal basis for $\mathbb{R}^n$ such that $v_1, \dots, v_m$ is an orthonormal basis for the image of F. If $\alpha = 0$ then clearly it is in the image of F and we are done. If $m = n$ we are done, since the image of F is the entire space. If $m < n$ we can define the functional $v_j : \mathbb{R}^n \to \mathbb{R}$ by $\sum a_i v_i \mapsto a_j$, for $j > m$. If we now take $\widetilde{v}_j$, the same functional defined in terms of the standard basis on $\mathbb{R}^n$, we will have a functional $\widetilde{v}_j$ which is zero on the image of F. Then we know that $\widetilde{v}_j \alpha = 0$, so the j-th coordinate of α in the v-basis was 0. However, this is the case for all $j > m$, so α is in the image of F and we are done.

5.14.16 Let $N_i = \ker g_i$. Each of these has codimension 1 since each g is a functional. Let $N = \bigcap_{i=1}^n N_i$, the joint kernel of all of the g's. This has codimension at most n, so we know that X/N is a finite-dimensional vector space. Each of the g_i is well-defined on X/N since $N \subset N_i$ for all i. The given conditions also imply that f is well-defined on X/N. So now we just need to show the desired statement for X/N, which is a finite-dimensional vector space of dimension at most n.

For each $g_i \in (X/N)^*$ there exists a $y_i \in X/N$ such that $g_i(x) = \langle x, y_i \rangle$ for all x. There also exists such a y for x. Thus showing that f is in the span of the g_i is equivalent to showing that y is in the span of the y_i. However, we know that $\dim(X/N)^* = \dim(X/N) \leq n$ and we have $n+1$ vectors, so there are constants $a_1, \dots, a_n, a$ such that

$$\sum_{i=1}^n a_i y_i + ay = 0.$$

However, since the g_i are linearly independent, the y_i are linearly independent and so $a \neq 0$. Therefore, if we let $\lambda_i = -a_i/a$ we have the desired result.