Solutions to 7.10, 7.20, and 8.8

Michael McElroy

May 22, 2005

Problem 7.10 (Luenberger)

> Consider the effect of adding an additional source of constant income in the estate-planning problem (p. 182-3 of Luenberger)

Solution to 7.10

Same utility function as before, $\int_0^T \mathrm{e}^{-\beta t}\, U[r(t)]\, \mathrm{d}t$. If $x(t)$ is the total captial at time t, then we have

$$\dot{x} = \alpha x(t) - r(t) + Q,$$

where α is interest rate, r is consumption rate, and Q is our constant source of additional income (trust-fund, lottery payouts, etc). Hence we wwant to maximize

$$\int_0^T \mathrm{e}^{-\beta t}\, U[\alpha x(t) - \dot{x} + Q]\, \mathrm{d}t.$$

The Euler-Lagrange equation for maximizing $\int f(x, \dot{x}) \mathrm{d}t$ is $\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial \dot{x}}$. Hence our Euler-Lagrange equation is

$$\alpha \mathrm{e}^{-\beta t}\, U'[\alpha x(t) - \dot{x} + Q] + \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-\beta t}\, U'[\alpha x(t) - \dot{x} + Q] = 0 \implies$$
$$\frac{\mathrm{d}}{\mathrm{d}t} U'[\alpha x(t) - \dot{x} + Q] = (\beta - \alpha) U'[\alpha x(t) - \dot{x} + Q].$$

So nothing new yet. Integrating yields

$$U'[r(t)] = U'[r(0)]\, \mathrm{e}^{(\beta - \alpha)t}.$$

Choose $U[r] = 2\sqrt{r}$, so that $U' = 1/sqrtr$ and

$$\alpha x(t) - \dot{x} + Q = r(t) = r(0) \mathrm{e}^{2(\alpha - \beta)t}.$$

So now we have Q added to $r(t)$.

$$\dot{x} = \alpha x - r(0) \mathrm{e}^{(\beta - \alpha)t} + Q.$$

This adds a term to $x(t)$:

$$x(t) = \mathrm{e}^{\alpha t} x(0) + \frac{r(0)}{\alpha - 2\beta} \left[\mathrm{e}^{\alpha t} - \mathrm{e}^{2(\alpha - \beta)t} \right] + Q \frac{\mathrm{e}^{\alpha t} - 1}{\alpha}.$$

So we can solve for $r(0)$ by requiring $x(t) = 0$. □

Problem 7.10 (Luenberger)

Consider the effect of adding an additional source of constant income in the estate-planning problem (p. 182-3 of Luenberger)

Solution to 7.10

Same utility function as before, $\int_0^T e^{-\beta t} U[r(t)]\,dt$. If $x(t)$ is the total captial at time t, then we have

$$\dot{x} = \alpha x(t) - r(t) + Q,$$

where α is interest rate, r is consumption rate, and Q is our constant source of additional income (trust-fund, lottery payouts, etc). Hence we wwant to maximize

$$\int_0^T e^{-\beta t} U[\alpha x(t) - \dot{x} + Q]\,dt.$$

The Euler-Lagrange equation for maximizing $\int f(x, \dot{x})dt$ is $\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}$. Hence our Euler-Lagrange equation is

$$\alpha e^{-\beta t} U'[\alpha x(t) - \dot{x} + Q] + \frac{d}{dt} e^{-\beta t} U'[\alpha x(t) - \dot{x} + Q] = 0 \implies$$
$$\frac{d}{dt} U'[\alpha x(t) - \dot{x} + Q] = (\beta - \alpha) U'[\alpha x(t) - \dot{x} + Q].$$

So nothing new yet. Integrating yields

$$U'[r(t)] = U'[r(0)]\, e^{(\beta-\alpha)t}.$$

Choose $U[r] = 2\sqrt{r}$, so that $U' = 1/sqrtr$ and

$$\alpha x(t) - \dot{x} + Q = r(t) = r(0) e^{2(\alpha-\beta)t}.$$

So now we have Q added to $r(t)$.

$$\dot{x} = \alpha x - r(0) e^{(\beta-\alpha)t} + Q.$$

This adds a term to $x(t)$:

$$x(t) = e^{\alpha t} x(0) + \frac{r(0)}{\alpha - 2\beta}\left[e^{\alpha t} - e^{2(\alpha-\beta)t}\right] + Q\frac{e^{\alpha t} - 1}{\alpha}.$$

So we can solve for $r(0)$ by requiring $x(t) = 0$. □

Problem 7.20 (Luenberger)

See Luenberger p. 211.

Solution to 7.20

We can return to the dealer any unsold goods, so we need not consider the cost of purchase into our profit equation. We only consider raw profitability. It is obvious off the bat that we should 400 hot dogs and 200 ice cream, but let's work through the Fenchel duality analysis to show how it works. We work straight from example 1 on pp. 202-203.

Let x_1 be the amount cost of hot dogs purchased and x_2 be the cost of ice cream purchased. So the numbers of hot dogs and ice cream are $5x_1$ and $10x_2$. Let $g_1(x_1)$ be the expected profit of hot dogs and let $g_2(x_2)$ be the expected profit of ice cream:

$$g_1(x_1) = \frac{1}{2} 0.3\left[\min(400, 5x_1) + \min(200, 5x_1)\right] \quad \text{and} \quad g_2(x_2) = \frac{1}{2} 0.1\left[\min(1000, 10x_2) + \min(200, 10x_2)\right].$$

Figure 1: Figures of the profit functions and their conjugate functionals.

Our primal problem is

$$\max_{\substack{x_1,x_2\geq 0\\ x_1+x_2=100}} g_1(x_1)+g_2(x_2).$$

So set D to be the positive orthant, f to be the zero functional, and $C \equiv \{(x_1,x_2) \mid x_1+x_2=100\}$. The conjugate functional for f is

$$f^*(y_1,y_2) = \sup_{x_1+x_2=100} y_1x_1+y_2x_2,$$

which is finite only if $y_1 = y_2$. Note that in $f^*(y_1,y_2)$ we only assume that the sum of x_1 and x_2 equals 100, while not assuming that $x_1, x_2 \geq 0$. Anyway, if $y_1, y_2 = \lambda$, then $f^*(y_1,y_2) = 100\lambda$. Define $C^* \equiv \{(y_1,y_2) \mid y_1 = y_2\}$ and $f^*(\lambda,\lambda) = 100\lambda$.

Now we define the conjugate functionals $g_i^*(x_i)$ of the profit functions $g_i(x_i)$:

$$g_j^*(\lambda) \equiv \inf_{x_j\geq 0} [x_j\lambda - g_j(x_j)]. \quad (1)$$

Now it is helpful to actually draw out the profit functions and their conjugate functionals. The dual problem is

$$\min_{\lambda} [100\lambda - g_1^*(\lambda) - g_2^*(\lambda)]. \quad (2)$$

Since the conjugate functionals are concave, we only need to evaluate this function at the values of λ where the conjugate functionals change slope and at 0, i.e. $\lambda \in \{0,\ 1/2,\ 3/4,\ 1,\ 3/2\}$. This equals

$$\min\{0+90+60,\ 50+50+10,\ 75+5+30,\ 100+20+0,\ 150+0+0\} = 110.$$

Hence 110 is our maximal profit. The dual problem (2) is minimized for $\lambda \in [1/2, 3/4]$, a continuous interval. Coming back to our graphs of the conjugate functonals, we find that for $\lambda = 1/2$, the values of (x_1,x_2) that minimize the right hand side of (1) are $(x_1,x_2) \in \{80\} \times [20,100]$. For $\lambda = 3/4$, we have $(x_1,x_2) \in [40,80] \times \{20\}$, and for $1/2 < \lambda < 3/4$ we have $(x_1,x_2) = (80,20)$.

□

Problem 8.8 (Luenberger)

Let f be a functional on a normed space X. $x_0^* \in X^*$ is a subgradient of f at x_0 if

$$\forall x \in X, \quad f(x) - f(x_0) \geq \langle x - x_0,\ x_0^* \rangle.$$

Show that if f has a gradient at x_0, then any subgradient equals the gradient at $x = x_0$.

Solution to 8.8

Assume that f has a gradient ∇f at some given point $x_0 \in X$. Assume also that a subgradient $z^* \in X^*$ of f at x_0 exists. For any $h \in X$ and $x \equiv h + x_0$ we have

$$f(x_0+h) - f(x_0) \geq \langle h,\ z^* \rangle.$$

Using the gradient's definition in terms of Fréchet derivatives, we have

$$\langle h,\ \nabla f\rangle = \lim_{\alpha\to 0}\frac{f(x_0+\alpha h)-f(x_0)}{\alpha} \geq \frac{\langle \alpha h,\ z^*\rangle}{\alpha} = \langle h,\ z^*\rangle.$$

So for any $x \in X$, we have

$$\langle x,\ \nabla f\rangle \geq \langle x,\ z^*\rangle.$$

If $z^* \neq \nabla f$, then we may choose $h \in X$ such that $\langle h,\ z^*\rangle \neq \langle h,\ \nabla f\rangle$. Define $D \equiv \langle h,\ \nabla f\rangle - \langle h,\ z^*\rangle > 0$. Using the definition of a limit, for $\epsilon = D/2 > 0$, there exists $\delta > 0$ such that

$$0 \geq \alpha < \delta \implies \left\| \langle -h,\ \nabla f\rangle - \frac{f(x_0-\alpha h)-f(x_0)}{\alpha} \right\| < \epsilon \implies \| \langle -\alpha h,\ \nabla f\rangle - [f(x_0-\alpha h)-f(x_0)] \| < \frac{D}{2}\alpha.$$

We also know that $\langle -\alpha h,\ \nabla f\rangle - \langle -\alpha h,\ z^*\rangle = -\alpha D$. Thus

$$[f(x_0-\alpha h)-f(x_0)] - \langle -\alpha h,\ z^*\rangle \in (-\alpha\frac{3}{2}D, -\alpha\frac{1}{2}D) < 0,$$

contradicting the definition of the subgradient. Hence $z^* = \nabla f$.

□