ANSWER TO SELECTED PROBLEMS IN HOMEWORK SETS 5

partial answer to the additional problem:

Let $L(x) = ax$, $L_1(x) = bx$, where both $a > 1$ and $b > 1$. Notice that both L and L_1 have a repelling fixed point at $x = 0$. We want to construct a homeomorphism $h : R \to R$ such that $h \circ L = L_1 \circ h$, i.e. $h(ax) = bh(x)$ for all $x \in R$.

We see that a fundamental domain for L is $(1, a] \cup [-a, -1)$. We know $L^n(x) = a^n x$. L^n maps $(1, a]$ onto $(a^n, a^{n+1}]$ and $[-a, -1)$ onto $[-a^{n+1}, -a^n)$. Moreover, $L^n(1, a] = (a^n, a^{n+1}]$ and $L^m[-a, -1) = [-a^{m+1}, -a^m)$, $n, m \in Z$ are mutually disjoint and

$$\cup_{n=-\infty}^{\infty}(a^n, a^{n+1}] = (0, \infty), \qquad \cup_{n=-\infty}^{\infty}[-a^{n+1}, -a^n) = (-\infty, 0).$$

We can take a fundamental domain for L_1 as $(1, b] \cup [-b, -1)$.

Now define $h : (1, a] \to (1, b]$ by the linear function

$$h(x) = \frac{b-1}{a-1}(x-1) + 1.$$

This function h is a homeomorphism from $(1, a] \to (1, b]$, and $h(1) = 1$, $h(a) = b$. Now from $h \circ L = L_1 \circ h$, we have $h \circ L^n = L_1^n \circ h$, or $h = L_1^n \circ h \circ L^{-n}$. Notice that for $x \in (a^n, a^{n+1}]$, $L^{-n}(x) \in (1, a]$. For $n \in Z$, we define

$$h(x) = L_1^n \circ h \circ L^{-n}(x) \quad \text{for } x \in (a^n, a^{n+1}]$$

This defines $h(x)$ for all $x \in (0, \infty)$. It is clear that this function $h : (0, \infty) \to (0, \infty)$ is a homeomorphism. Likewise, we define $h : [-a, -1) \to [-b, -1)$ by the linear function taking $h(-1) = -1$ and $h(-a) = -b$; and define

$$h(x) = L_1^n \circ h \circ L^{-n}(x) \quad \text{for } x \in [-a^{n+1}, -a^n)$$

This defines $h : (-\infty, 0) \to (-\infty, 0)$. Since h maps fixed points to fixed points, and 0 is the only fixed point for both L and L_1, we must define

$$h(0) = 0$$

Since $\lim_{n\to-\infty} h(a^n) = \lim_{n\to-\infty} b^n = 0$, $\lim_{n\to-\infty} h(-a^n) = -\lim_{n\to-\infty} b^n = 0$, h is continuous at 0. This finishs the construction of the homeomorphism $h : R \to R$ such that $h \circ L = L_1 \circ h$. QED.

§1.9:5 We want to show that $F_4(x) = 4x(1-x)$ is not structurally stable. Consider

$$g_\epsilon(x) = 4x(1-x) + \epsilon,$$

where $\epsilon > 0$. Notice that $d_r(F_4, g) = \epsilon$, for all r. Solving $g_\epsilon(x) = x$ we find g_ϵ has two fixed points:

$$\alpha = \frac{3 - \sqrt{9+16\epsilon}}{8}, \quad \beta = \frac{3 + \sqrt{9+16\epsilon}}{8}.$$

Now $g_\epsilon(1/2) = 1 + \epsilon$, and $g_\epsilon^2(1/2) = g_\epsilon(1+\epsilon) = -4(1+\epsilon)\epsilon$. Since

$$\alpha = \frac{(3 - \sqrt{9+16\epsilon})(3 + \sqrt{9+16\epsilon})}{8(3 + \sqrt{9+16\epsilon})} = \frac{-2\epsilon}{3 + \sqrt{9+16\epsilon}} > -\epsilon$$

we have

$$g_\epsilon^2(1/2) < \alpha$$

Graphic analysis shows that for $x < \alpha$, we have $g_\epsilon^n(x) \to -\infty$ monotonically as $n \to \infty$. Now we want to show that g_ϵ is not topologically conjugate to F_4. If not, we assume that there is a homeomorphism $h : R \to R$ such that $h \circ F_4 = g_\epsilon \circ h$. h is either increasing or decreasing. Since F_4 has two fixed points: 0 and 1, and h takes fixed points of F_4 to fixed points of g_ϵ, we have that $h : [0,1] \to [\alpha, \beta]$ is a homeomorphism. In particular, $h^{-1}(1/2) \in [0,1]$, since $1/2 \in [\alpha, \beta]$. Now from $h \circ F_4 = g_\epsilon \circ h$, we have $h \circ F_4^n = g_\epsilon^n \circ h$. So

$$g_\epsilon^n(1/2) = h \circ F_4^n \circ h^{-1}(1/2) \tag{1}$$

Notice that the left hand side of (1) goes to $-\infty$ as $n \to \infty$ while the right hand side of (1) stays in interval $[\alpha, \beta]$ for all $n.Thisisacontradiction.Thereforeg_\epsilon$ and F_4 are not topologically conjugate. QED.