Math118. Problem set **1**

The Henon maps.

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In class we have been concentrating on maps defined on an interval. The point of this problem set is to introduce a family of maps of the plane into itself which depend on two parameters, a and b. The maps are easy to write down; nevertheless they exhibit much of the interesting phenomena that can arise in general. The Henon maps $\mathbf{h} = \mathbf{h}_{a,b}$ are:

$$\mathbf{h}(x, y) = (a - x^2 + by, x).$$

Thus the new y is just the old x and the new x involves the old x and y. It is non-linear, but the non-linearity appears mild, involving just the term x^2.

1. Write an m.file to plot the long term behavior of the iterates of a point under the Henon maps. So this file should call for the input of a and b together with the input of an initial condition and the number k of iterates to be plotted. It should then successively apply the Henon map with these parameters for say 100 times, and then plot the next k iterates.

The command "plot(x,y)" where x and y are numbers places a blue dot at the point whose coordinates are (x, y). An isolated point such as this is barely visible on the screen and may not print out. The command "plot(x,y,'.')" prints a blue dot which will print out, but is still barely visible. The command plot(x,y,'*') prints a blue star at the point (x, y) and should be used when few points are expected. If x and y are are row vectors, the command "plot(x,y,'.')" will place a (blue) dot at each of the points $(x(1), y(1)), (x(2), y(2)$, etc. Similarly for "plot(x,y,'*')". The command "plot(x,y,'.')" should be used when many points are expected as in parts 3) 4) and 5) of problem **8**. (Remember to put a semicolon at the end of every line whose output you do not want to appear on the screen.)

2. Fix $b = -.3$. With initial condition $(0, 0)$ and $a = 1.2$ what is the limiting set? With the same initial condition and $a = 1.3$ what is the limiting set? You might want to use zoom to aid in seeing what is going on. Where does the transition between these two types of limiting sets appear to occur? You may need to examine a large number of iterates.

For a partial discussion of the above "experimental" results solve the following two problems:

3. Show that $\mathbf{h}_{a,b}$ has no fixed points if $a < -\frac{1}{4}(1-b)^2$, has one fixed point if $a = -\frac{1}{4}(1-b)^2$ and has two fixed points if $a > -\frac{1}{4}(1-b)^2$. Locate the fixed points when they exist. (Hint: use the quadratic formula and the fact that $x = y$ at a fixed point of a Henon map.)

4. Show that $\mathbf{h}_{a,b}$ has a (strictly) period two orbit if and only if $a > \frac{3}{4}(1-b)^2$. Determine this orbit when it exists. (Hint: Write out the equations for a fixed point of $\mathbf{h}^{\circ 2}$. Solve the equation in y to give y as a function of x. Substitute this into the equation for x. This gives a degree four equation in x, i.e. an equation of the form $P(x) = 0$ where P is a degree four polynomial whose coefficients depend on a and b. But we know that a fixed point is automatically a point of period two. So we know that P must factor as $P = Q_1 Q_2$ where $Q_1(x) = 0$ is the quadratic in Problem **3**. So find Q_2.)

5. For $b = -.3$ at which value of a does the period two orbit appear?

Problems **3-5** only give a partial explanation of the phenomena observed in Problem **1**, since, in addition to locating the fixed points of the periodic points, we must discuss whether or not they are attractors. In one dimension this depended on the derivative of the map. In more than one dimension, the derivative gets replaced by the **Jacobian matrix**. Recall that if $\mathbf{f}$ is a map from m-dimensional space to n-dimensional space, then $\mathbf{f}$ is given by n functions of m variables, and its Jacobian matrix $D\mathbf{f}$ is the matrix of its partial derivatives. For example, if

$$\mathbf{f} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

is a map of the plane to the plane then

$$D\mathbf{f}(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Here each of the entries in the matrix on the right are themselves functions of x and y. In our case

$$D\mathbf{h}(x,y) = \begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}$$

does not depend on y or on a.

Recall that the eigenvalues of a square matrix M are the roots of the *characteristic polynomial*

$$\det(\lambda I - M).$$

For a two by two matrix this is a quadratic polynomial. In general, if you type in a (numerical) square matrix M into matlab, the command "eig(M)" will produce the eigenvalues. But for two by two matrices we can use the quadratic formula. Thus the eigenvalues of $D\mathbf{h}$ are

$$\lambda_{\pm} := -x \pm \sqrt{x^2 + b}.$$

This can be checked from the fact that the sum of the two roots is $-2x = \mathrm{tr} D\mathbf{h}$ while their product is $-b = \det(D\mathbf{h})$.

Now a fixed point is an attractor if and only if *all* the eigenvalues of the Jacobian matrix at the fixed point have absolute value strictly less than one. (We will discuss this in general in class.) The fixed point(s) of the Henon map satisfy $x = y$ and

$$x = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2}$$

if the square root is not imaginary. When $a = -\frac{1}{4}(1-b)^2$ the fixed points first appear with

$$x_+ = x_- = \frac{b-1}{2}$$

we have

$$\lambda_+ = 1, \quad \lambda_- = -b.$$

Let us study the variation of behavior as a varies but b is fixed and

$$|b| < 1.$$

6. Show that $x = \frac{b-1}{2}$ is the only value of x for which $\lambda_+(x) = 1$ and that $\lambda_+(x)$ is a decreasing function of x near $x = \frac{b-1}{2}$. Also show that if the eigenvalues are complex (at some value of x) then they must satisfy. $|\lambda_+| = |\lambda_-| = |b|^{\frac{1}{2}}$. If we write $x_\pm(a) = \frac{(-(1-b)\pm\sqrt{(1-b)^2+4a}}{2}$ for the x coordinate of the fixed points corresponding to a, we see that $x_-(a)$ decreases as a increases from $-\frac{1}{4}(1-b)^2$. Thus $\lambda_+(x_-(a))$ increases as a function of a and so is > 1 for a greatere than but near to $-\frac{1}{4}(1-b)^2$. Conclude that the fixed points $(x_-(a), x_-(a))$ are all unstable, and hence invisible in the computer experiment. By a similar argument, show that for $a = \frac{3}{4}(1-b)^2$ we have

$$\lambda_-(x_+(a)) = -1$$

and that the fixed points $(x_+(a), x_+(a))$ are stable, i.e. are attractors, for

$$-\frac{1}{4}(1-b)^2 < a < \frac{3}{4}(1-b)^2.$$

At $a = \frac{3}{4}(1-b)^2$ the fixed point $(x_+(a), x_+(a))$ undergoes a period doubling bifurcation. Indeed, the theorem we proved in class about period doubling bifurcations for one dimensional maps has a higher dimensional analogue: whenever a single eigenvalue at a fixed point passes through -1 (under hypotheses similar to the one dimensional case) and all the other eigenvalues are unequal to -1 a period doubling bifurcation occurs - the stable fixed point becomes unstable and a stable = attractive periodic orbit of period two is created. The proof is obtained by using the implicit function theorem to reduce the general theorem to the one dimensional case. We may or may not have time to do this in class.

7. Show that if (x_1, y_1) and (x_2, y_2) is a period two orbit of the Henon map then

$$x_1 + y_1 = x_2 + y_2 = x_1 + x_2 = y_1 + y_2 = 1 - b.$$

In particular all period two orbits lie on the line $x + y = 1 - b$.

Do not get the impression from the above problems that the limit sets of the Henon map are simple. Quite the contrary.

8. Fix $b = 0.4$. Run (and print out) your m.file for the following values of a and initial conditions $(0, 0)$:

1. $a = .9$. You should see a period four orbit.
2. $a = 0.988$. You should see a period sixteen orbit.
3. $a = 1$. You should see a four piece "attractor".
4. $a = 1.045$. A two piece attractor with points alternating between the pieces.
5. $a = 1.2$. The two pieces have merged to form a one-piece attractor.