Math 118, Problem set 3

Singer's theorem.

Posted March 8, 2001. Due March 23, 2001.

1. Run logisticlimit for R in the range (3.625,3.635). You will see a "window" with six horizontal curves. Verify that this represents a single attractive orbit of period six, and not, for example two attracting three cycles.

The goal of the next few problems is to prove a general theorem which implies that the logistic map has at most one attracting cycle for any $1 < \mu < 4$ on the unit interval $I = [0,1]$. Recall that we defined the basin of attraction of a fixed point p of a map f to be the set of all x such that $f^j(x) \to p$ as $j \to \infty$. If p is a periodic point of f of period n, then we define the basin of attraction of the corresponding periodic orbit $O(p) := \{p, f(p), \dots, f^{\circ(n-1)}(p)\}$ to be the basin of attraction of $g = f^{\circ n}$. We need a slightly more delicate concept: Suppose p is a fixed point of f whose basin of attraction contains an open interval containing p. (This will always be the case if $|f'(p)| < 1$.) We define the **immediate basin of attraction** of p to be the maximal interval J containing p and contained in the basin of attraction of p. Suppose that J is the immediate basin of attraction of p. Then $f(J) \subset J$ since if $f^{\circ n}(x) \to p$ then so does $f^{\circ n}(f(x))$ and $f(x)$ is connected to p by a whole curve of such points. Suppose that f is defined on an interval $I = [a,b]$. Then the left hand end point of J might be a. But if the left hand end point of J is some number $c > a$, then $c \notin J$. Because if it were, then some power of f would map c into a neighborhood of p consisting of points all converging to p, and hence some small neighborhood of c would be mapped into this same neighborhood of p by continuity. So either one or the other end points of J consists of an end point of I, or J is an open interval, $J = (c,d)$. In the case of the logistic map, the point 0 is a fixed point, and so is not in the basin of attraction of any periodic orbit for $\mu > 1$, and neither is the point 1 which gets mapped to 0 under one application of the logistic map.

If p is a periodic point with orbit $O(p) = \{p, f(p), \dots, f^{\circ(n-1)}(p)\}$ we define the immediate basin of attraction of $O(p)$ to be the union of the immediate basins of attraction of $f^{\circ n}$ of each of the points of $O(p)$.

Suppose that $f : I \to I$ has three continuous derivatives. The **Schwarzian derivative** of f defined at points where $f'(x) \neq 0$ is defined as

$$(Sf)(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{1}$$

2. Compute the Schwarzian derivative of the logistic function (at all $x \neq \frac{1}{2}$ which is the unique critical point).

3. Verify that if f and g have three continuous derivatives then

$$s(g \circ f) = (Sg)(f(x)) \cdot (f'(x))^2 + (Sf)(x). \tag{2}$$

(In fancy language this equation says that the Schwarzian derivative is a cocycle for the group of diffeomorphisms acting on the space of quadratic differentials. We will not make use of this fancy language.) Conclude from (2) that if $S(f) < 0$ and $S(g) < 0$ at all points then $S(f \circ g) < 0$. In particular, if $S(f) < 0$ at all points then $S(f^{\circ n}) < 0$.

4. Show that if $f(x) = ax + b$ or $f(x) = 1/x$ then $S(f) \equiv 0$. Conclude that if $f(x) = (ax+b)/(cx+d)$ then $S(f) \equiv 0$. Can you prove the converse: that if $S(f) \equiv 0$ then f is of the form $f(x) = (ax+b)/(cx+d)$? [Hint: If $f' > 0$ give an alternative expression for $S(f)$ in terms of the second derivative of $(f')^{-\frac{1}{2}}$.]

For the next few problems we make the standard assumption that

$$S(f) < 0$$

at all points where $f'(x) \neq 0$. Recall that a point x where $f'(x) = 0$ is called a critical point of f.

5. Show that if $f''(y) = 0$ then $f'(y)$ and $f'''(y)$ have opposite signs. Conclude that if y is a interior point, then either it is a local maximum for f' and then $f'(y) > 0$ or it is a local minimum for f' with $f'(y) < 0$. Conclude that if f' does not vanish on the interval $[c, d]$ then the minimum of $|f'|$ is taken on at one or the other of the end points. More precisely, show that

$$|f'(x)| > \min\{|f'(c)|, |f'(d)|\} \quad \forall\, c < x < d.$$

Let p be a point of period n of f and suppose that there is an open interval containing p and lying in the basin of $f^{\circ n}$ at its fixed point p. Let T be the immediate basin of $f^{\circ n}$ at p, so T is one of the intervals in the immediate basin of $O(p)$. Suppose that T does not contain one of the boundary points of I. So we know that

$$f(T) \subset T \tag{3}$$

and

$$f(\partial T) \subset \partial T \tag{4}$$

where ∂T denotes the (two points of the) boundary of T.

6. Show that if $f^{\circ n}$ has a critical point in T then f has a critical point in the immediate basin of attraction of $O(p)$.

We wish to prove:

Theorem 1 [**D. Singer.**] *Suppose that $f : I \to I$ is a function with negative Schwarzian derivative (where $f'(x) \neq 0$) and that p is a point of period n for f such that $O(p)$ has a non-empty immediate basin of attraction as above. Then this immediate basin of attraction contains either an end point of I or a critical point of f.*

Proof. Suppose not. Then by Problem **6** the interval T does not contain a critical point of $f^{\circ n}$ where T is the immediate basin of attraction of $f^{\circ n}$ at p.

7. Suppose that $(f^{\circ n})'(x) > 0$ for all $x \in T$. Let $g = f^{\circ n}$. Show that $g(\partial T) = \partial T$ by (4). In fact $g(x) = x$ for $x \in \partial T$ since g is monotone increasing. Show that $g'(x) \geq 1$ for $x \in \partial T$ and then conclude that $g'(p) > 1$ contradicting the existence of a non-empty immediate basin of attraction for g at p. If $(f^{\circ n})'(x) < 0$ for all $x \in T$ set $g = f^{\circ 2n}$ and repeat the above argument to conclude the proof of Singer's theorem. Conclude that the logistic map has at most one attractive cycle.

8. Write a Matlab program that calls for an input n and then plots the average of f, $f \circ L_4$, $f \circ L_4^2$, ..., $f \circ L_4^n$. Here you may choose f to be any (interesting) non-constant function. Attach a copy of the resulting graph. Interpret the result in terms of ergodic theory.

9. Write a Matlab program to demonstrate the fact that in a random walk of n steps, the probability $p(x)$ that the last place where the random walk crosses 0 is the fraction x of the way through is given by $p(x) = \frac{1}{\pi\sqrt{x(1-x)}}$. Your program should call for a length n for the random walks and a number r of random walks to test. It should figure out the last zero-crossing of each random walks, and list the number of random walks for which the last zero-crossing was at step k as a histogram. It should then compare this histogram with the function $p(x)$.