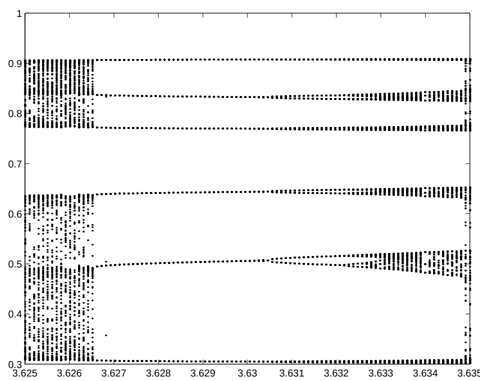
Solutions to Problem Set 3

April 2, 2001

1. Below is a plot of the logistic limit diagram for μ in the range $(3.625, 3.635)$. It is easy to verify numerically that the orbit in question is a period six orbit.



2. The answer is $-\frac{6}{(1-2x)^2}$, independent of the choice of μ.

3. Direct verification. The correct statement, as everyone managed to figure out, is that $S(g \circ f) = (Sg)(f(x))(f^{'}(x))^2 + (Sf)(x)$.

If $S(f) < 0$ and $S(g) < 0$ for all x, then both terms on the right-hand side are negative, so $S(g \circ f) < 0$ for all x. By induction, if $S(f) < 0$ then $S(f^{\circ n}) < 0$.

4. It is easy to show that the Schwarzian derivatives of $ax + b$ and $\frac{1}{x}$ are 0. Any function $f(x) = \frac{ax+b}{cx+d}$ is a composition of functions of the form $ax + b$ and $\frac{1}{x}$. By the composition law, one checks that the Schwarzian derivative of $f(x) = \frac{ax+b}{cx+d}$ is 0 as well.

Conversely, suppose $Sf(x)$ is identically 0. Then $f^{'} \neq 0$ for all x, so $f^{'}$ does not change sign. Assume $f^{'} > 0$. One checks that the second derivative of $f'(x)^{\frac{1}{2}}$ is $-\frac{1}{2}(f'(x))^{-\frac{1}{2}}(Sf)(x)$, so $Sf(x) = -2(f^{'}(x))^{\frac{1}{2}}((f^{'}(x))^{-\frac{1}{2}})^{''}$. For Sf to be 0, we must have either $f^{'} = 0$, in which case f is constant, or $((f^{'}(x))^{-\frac{1}{2}})^{''} = 0$, in which case $(f^{'}(x))^{-\frac{1}{2}}$ is linear, $f^{'}(x)$ is of the form $\frac{1}{(ax+b)^2}$, and $f(x)$ is of the form $\frac{Ax+B}{Cx+D}$.

5. If $f^{''}(y) = 0$ and $f^{'}(y) \neq 0$, then $\frac{f^{'''}(y)}{f^{'}(y)} - \frac{3}{2}(\frac{f^{''}(y)}{f^{'}(y)})^2 = (Sf)(y) < 0$. Hence $\frac{f^{'''}(y)}{f^{'}(y)} < 0$, so $f^{'}(y)$ and $f^{'''}(y)$ have opposite signs.

If $f^{''}(y) = 0$ and $f^{'''}(y) < 0$, then y is a local maximum for $f^{'}$ by the second derivative test. In this case $f^{'}(y)$ is positive. Similarly if $f^{'''}(y) > 0$ then y is a local minimum for $f^{'}$ and $f^{'}(y) < 0$.

If $f^{'}$ does not vanish on $[c, d]$, then without loss of generality assume $f^{'}$ is positive on $[c, d]$. Suppose $f^{'}$ achieves a global minimum at some point x in (c, d). Since x is an interior point, x is a local minimum and thus $f^{''}(x) = 0$. But since $f^{'}$ is positive on $[c, d]$, we must have that $f^{'''}(x) < 0$, i.e. x is a local maximum for $f^{'}$, a contradiction. Hence $f^{'}$ achieves no global minimum in the interior of $[c, d]$, so its minima must occur at the endpoints. A similar result holds if $f^{'}$ is negative.

6. Let U denote the immediate basin of attraction of $O(p)$. Suppose that f has no critical points in U. Then Sf is negative on all of U. Since f maps U to U, $Sf^{\circ n}$ is negative on all of U. Hence $f^{\circ n}$ has no critical points in U and thus has no critical points in T, a contradiction.

7. Suppose that the conclusion of Singer's theorem does not hold. Then by problem 6, the interval T doesnot contain a critical point of g.

Assume that $g^{'} > 0$ on all of T.

Since $g(\partial T) = \partial T$ and g is monotonically increasing, we must have $g(x) = x$ for both x in ∂T. If $g'(x) < 1$ for either x in ∂T, then x is an attracting fixed point for g and thus has a nontrivial basin of attraction V containing x. But then V and T intersect. Now the immediate basins of attraction for two distinct fixed points cannot intersect, so we have a contradiction. Hence $g'(x) \geq 1$ for both x in ∂T. By problem 5, $g^{'}(p) > 1$, contradicting the fact that p is an attracting fixed point. Hence we cannot have $g^{'} > 0$ on all of T. The case where $g^{'} < 0$ is handled by setting $h = g^{\circ 2}$. Then $h^{'}(x) = g^{'}(g(x))g^{'}(x)$ for all x in T. Since $g(x)$ is in T, we have that both $g^{'}(g(x))$ and $g^{'}(x)$ are negative, so $h^{'}(x)$ is positive, and the rest of the proof proceeds as above.

For $\mu > 1$, the logistic function L_μ has the unique critical point $\frac{1}{2}$, and the fixed point 0 is repelling. Hence any nontrivial basin of attraction must contain the critical point $\frac{1}{2}$. Since basins of attractions for different attracting periodic points are disjoint, there is at most one nontrivial basin of attraction. If $\mu > 4$, there are in fact no nontrivial basins of attraction, as the invariant set Λ for L_μ (the set for which the iterates of L_μ remain bounded) contains no intervals (i.e., it is totally disconnected).

8. Here's the program I wrote:

```
% This program demonstrates Birkhoff's ergodic theorem.
% It asks for an integer n, then plots the average of
% f, f \circ L_4, f \circ L_4^2, etc.
clear;
close all;
n=input('Enter n, the number of iterations to average: ');
```

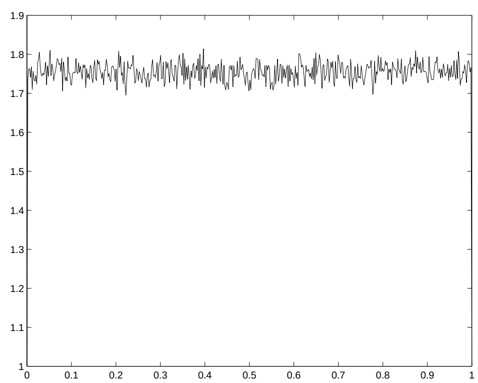
```
k=input('Enter the number of points on the interval [0,1] to graph: ');
x = linspace(0,1,k);
A = zeros(n+1,k);
A(1,:) = exp(x);
B = x;
for i=1:n
B = logistic(B,4);
A(i+1,:) = exp(B);
end

C=zeros(1,k);
for i=1:n+1
C = C + A(i,:);
end

C = C./(n+1);

plot(x,C)
```

Below is a graph where $f(x) = e^x$, $n = 500$ and we take 500 sample points in the interval $[0, 1]$.



Birkhoff's ergodic theorem says that as n approaches infinity, the limiting function should approach $\int f\mu$ pointwise except on a set of μ-measure zero. Note that $\int f\mu$ is a constant, independent of x. The blips in the graph (roughly) represent points in the set of measure zero. The endpoints 0 and 1 are points in this set of measure zero, since the limiting function, evaluated at 0 and 1, is simply $e^0 = 1$.

9. Here's the program I wrote:

```
%Tests the arcsine law for random walks.
close all;
clear all;
N=input('Enter the length of the random walks: ');
```

```
r=input('Enter the number of random walks to try: ');

x=zeros(r,1);

for i=1:r
Q=zeros(1,N);
R=rand(1,N);
Q=Q+(R>.5);
P=2.*Q-1;
W = [0 cumsum(P)];
for j=1:N+1
if W(N+2-j) == 0
  x(i) = N+2-j;
  break;
end;
end;
end;
hist(x,(0:N));
hold on

y=linspace(1/N,1-(1/N),N-1);
p=2.*r ./ (N .* pi .* ((y .* (1-y)).^ .5));

%Normalize curve so that area under curve equals
%area in histogram bars
%Then multiply by 2 because only odd-numbered bins are
%filled, by parity

plot(linspace(1,N-1,N-1),p,'k')
```

Below is the result with 1000 random walks of 1000 steps each: