Math 118, Problem set 2

Chaos and Sarkovski's Theorem

Posted February 21, 2001. Due March 9, 2001.

1. The map $x \mapsto 2x^2 - 5x$ has 0 and 3 as fixed points. Find the points of period two and determined whether the period two orbit is attractive or repelling.

2. Draw the graphical iteration of the logistic map for $\mu = 3.97$ at one of its period three orbits through the points

$$A = 0.18670411963922, \quad B = 0.60282739465532, \quad C = 0.95052331182672.$$

In problems 3 through 7, we will prove that certain functions that have points of period 3 are chaotic in a sense to be made precise.

Let L denote the interval $[A, B]$ and R denote the interval $[B, C]$. Notice that $f(R) = L$ and $f(L) \supset R$, but there are some points in L (such as .5 and in fact a whole interval about .5) which get mapped out of the interval $[A, C]$. The next few problems deal with maps f defined on an interval $[A, C]$ which are *unimodal*, i.e. have a unique maximum and have period three with $A < B < C$ and $f(A) = B, f(B) = C, f(C) = A$ as is illustrated by the logistic function in the figure, with the maximum located in the interval $[A, B]$.

We want to concentrate our attention on points which stay in the interval $[A, C]$ under all iterations of f. The initial interval $[A, C]$ breaks up into two subintervals $L = [A, B]$ and $R = [B, C]$. We have $f(R) = L \cup R$ so we can break up the interval R into two subintervals, $R \cap f^{-1}(L)$ which we denote by RL and $R \cap f^{-1}(R)$ which we denote by RR. On the other hand, every point of L either gets mapped into R, and we shall denote the interval of such points by LR or gets mapped outside $[A, C]$ and we shall discard these points. So there may be a gap between the interval LR and the interval RR. The interval RR is immediately to the left of the interval RL. If we follow the next iteration, the interval LR breaks up into two subintervals LRR and LRL, then there may be a gap, followed by RRL and RRR (which constitute the break up of RR with no gap) then possible a gap followed by the interval RLR.

3. Draw the next stage, consisting of intervals with four choices of L and R as labels. Explain why in any "itinerary" of a point, an R can be followed by an R or an L, but an L must be followed by an R.

4. Let d denote the length of the interval RR. Let J be any interval of stage k+1 that ends in an R. So $J = S_1 S_2 \cdots S_k R$ where each S_i is either L or R (but no L is followed by an L). Thus J is the interval consisting of those points with $x \in S_1,\ f(x) \in S_2,\ f(f(x)) \in S_3$, etc. Show that J contains a pair of points , one in each of the subintervals $S_1 S_2 \cdots S_k RRL$ and $S_1 S_2 \cdots S_k RLR$ that eventually move a distance d apart from one another.

5. Show that inside J there are 3 subintervals of type $S_1 S_2 \cdots S_k R S_{k+2} S_{k+3}$. Explain why at least two of them have length less than one half the length of J.

6. Combine the two previous problems to show that a subinterval of the form $J = S_1 S_2 \cdots S_k R$ must contain a subinterval J_1 of the form

$$S_1 S_2 \cdots S_k R S_{k+2} S_{k+3} RLR$$

having the following property: Each point of J_1 has a neighboring point within distance length$(J)/2$ whose distance under pairwise iteration exceeds d.

7. Let $h = C - A$. Show that for any positive integer k there are 2^k disjoint intervals of length less than $2^{-k}h$ each whom has a neighbor at distance $< 2^{-k}h$ which moves apart eventually under iteration a distance at least d.

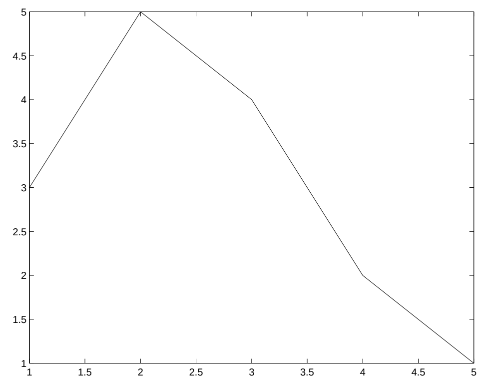
This last property is known as "sensitive dependence on initial conditions". We have shown that this is a property of any unimodal map with a period three orbit. This was first proved by Li and Yorke in a paper entitled "Period three implies chaos", *American Math. Monthly* **82** 985-992, introducing the word "chaos" to the mathematical (and general) public.

8. Write a Matlab script that calls for two values x_1 and x_2, a parameter μ and an integer N, then plots N iterates of x_1 and x_2 under the logistic function L_μ, and the difference between $L_\mu^{\circ n}(x_1)$ and $L_\mu^{\circ n}(x_2)$.

For sufficiently large values of μ, observe the sensitive dependence on initial conditions: if x_1 and x_2 are, say, 10^{-4} apart, then after a suitable number of iterations the plots will no longer resemble each other. (Of course, the Matlab calculations themselves are suspect after N iterations for some very large N. However, Matlab rounds off after 15 or so decimal places, whereas x_1 and x_2 differ in the 4th decimal place, so the divergence in the graphs is primarily due to the difference between x_1 and x_2 and not due to roundoff error.)

In the last two problems, we will construct functions that have points of period n but not of period m for certain integers n and m where n is the immediate successor of m in the Sarkovski ordering.

9. Consider the piecewise linear function $F : [1,5] \mapsto [1,5]$ drawn below. Here $F(1) = 3, F(2) = 5, F(3) = 4, F(4) = 2$ and $F(5) = 1$.

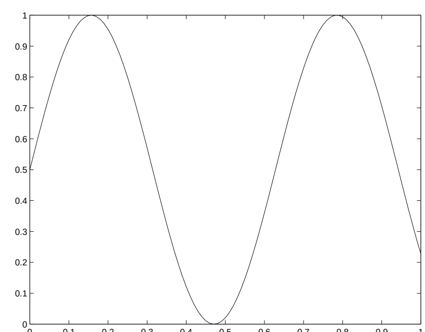


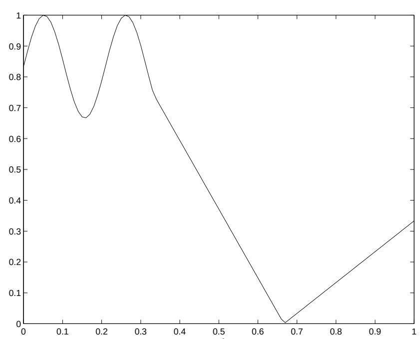
Clearly F has a point of period 5. Show that F has no point of period 3.

Hints: Show that $F^{\circ 3}$ has no fixed point in the intervals $[1,2]$, $[2,3]$, and $[4,5]$. Then show that the fixed point of $F^{\circ 3}$ in the interval $[3,4]$ is actually a fixed point of F.

10. If $f : I \mapsto I$, define a new function $Wf : I \mapsto I$ as follows:

$$Wf(x) = \begin{cases} \frac{f(3x)+2}{3} & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{(f(1)+2)(2-3x)}{3} & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ x - \frac{2}{3} & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$





Describe the periodic points of Wf. For each j, construct a function g_j that has points of periods 1, 2, 4, ..., 2^j but no point of period 2^{j+1}.