Math 118, Problem Set 4

The Sierpinski gasket

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The purpose of the problems in this problem set is to walk through part of the computation of the Hausdorff dimension of the Sierpinski gasket, and some related notions, all extracted from Edgar *Measure, Topology and Fractal Geometry.*

Recall that if (X, d) is a metric space and $A \subset X$, then diam A, the **diameter** of A, is defined to be sup $d(x, y)$ where the supremum is taken over all pairs of points x and y in X.

Next, define $\mathcal{H}^s_\epsilon$ to be inf $\Sigma_{A \in \mathcal{A}}(\text{diam} A)^s$, where the infimum is taken over all ϵ-covers $\mathcal{A}$ of A. An ϵ-cover $\mathcal{A}$ of A is any collection of sets which cover A, where each individual set has diameter at most ϵ.

One can show that as ϵ gets smaller, $\mathcal{H}^s_\epsilon$ gets larger. The s-dimensional **Hausdorff measure** $\mathcal{H}^s(A)$ of A is defined to be $\lim_{\epsilon \to 0} \mathcal{H}^s_\epsilon(A)$.

If A is a Borel set (all the sets we consider in this problem set will qualify), one can show that if $0 < s < t$, and $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$. Conversely, if $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$. Therefore, for a given set A there is a unique number $s_0 \in [0, \infty)$ such that $\mathcal{H}^s(A) = \infty$ for $s < s_0$ and $\mathcal{H}^s(A) = 0$ for $s > s_0$. This s_0 is the **Hausdorff dimension** of A.

Let F denote the set of (half)-infinite sequences from a 3 letter alphabet. Say the letters are L, U, and R. For each $0 < r < 1$ we can put a metric on F just as in the two letter case:

$$d_r(x, y) = r^{|\alpha|}, \quad \text{where } x = \alpha x',\ y = \alpha y'$$

and the initial letters of x' and y' are different. Here $|\alpha|$ is the length of α.

Note that the collection of sets $[\alpha]$ of all sequences with initial string α form a basis of the open sets. For reasons which will soon become clear, we will take $r = \frac{1}{2}$.

1. Assuming some of the facts about Hausdorff dimension quoted above, show that $(F, d_{\frac{1}{2}})$ has Hausdorff dimension $\log 3 / \log 2$.

The definition of Sierpinski gasket is that it is the compact subset of the unit square which is the unique fixed point in the set of compact subsets of the unit square under the map

$$K \mapsto T_L(K) \cup T_U(K) \cup T_R(K)$$

where the transformations T_L, T_U, T_R of the plane are given by

$$T_L : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y \end{pmatrix}, \quad T_U : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix}, \quad T_R : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y \end{pmatrix}.$$

Here x and y are **not** the usual Cartesian coordinates in the plane; instead, the unit vector $\mathbf{y}$ makes an angle $\frac{\pi}{3}$ with the unit vector $\mathbf{x}$. Thus the Sierpinski gasket is a subset of the unit equilateral triangle.

If we write the first bit in the binary expansion of the points in this set we have

1	J	
0	J	J
	0	1

Here J represents an equilateral triangle whose side lengths are $\frac{1}{2}$. (The diagram here shows a right triangle, but pretend that it's skewed over to represent an equilateral triangle.)

Notice that all points whose (only) binary expansion of the x and y coordinates have a 1 in the first position are excluded. At the next stage we have

11	J			
10	J	J		
01	J		J	
00	J	J	J	J
	00	01	10	11

2. Convince yourself that the Sierpinski gasket consists of all points $\begin{pmatrix} x \\ y \end{pmatrix}$ in the unit square where x and y have binary expansions such that a 1 never occurs in the same position in the expansion of x and of y.

Consider the map $h : F \to$ the unit square where the i-th letter in an element of F (which is a sequence of letters from the alphabet L, U, R) goes over into the i-th position in the binary expansion of x and y according to the rule

letter	bit of x	bit of y
L	0	0
U	0	1
R	1	0

so, for example,

$$h(LRLUU\ldots) = \begin{pmatrix} .01000\ldots \\ .00011\ldots \end{pmatrix}.$$

3 Show that

$$\|h(s) - h(t)\| \leq d_{\frac{1}{2}}(s,t)$$

where $\|\cdot\|$ denotes Euclidean distance in the plane, but in contrast to the Cantor set case, we do not have bounded decrease for the map h.

4. Conclude from **3** that the Hausdorff dimension of the Sierpinski gasket is $\leq \log 3/\log 2$.

The proof that the Hausdorff dimension of the Sierpinski gasket is exactly $\log 3/\log 2$ is a lot trickier. For a lot of extra credit you might want to try this.

Let $r_1, \ldots, r_n$ be real numbers satisfying $0 < r_i < 1$. We want to think of the r_i as the "ratio of contraction" of a transformation T_i of a metric space X, i.e.

$$d(T_i x, T_i y) \leq r_i d(x,y) \quad \forall\ x, y \in X.$$

Here we are considering the transformation on compact subsets of X given by

$$T(K) = T_1(K) \cup \cdots \cup T_n(K).$$

The collection $(r_1, \ldots, r_n)$ is called the **ratio list** of T. For example, for the Cantor set the ratio list is $(\frac{1}{3}, \frac{1}{3})$ while for Sierpinski gasket the ratio list is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

5. Show that for any ratio list there is a unique non-negative number s such that

$$r_1^s + \cdots + r_n^s = 1.$$

[Hint: Consider the function $f(s) := \sum_i r_i^s$ so $f(0) = n$ and $f(\infty) = 0$. Show that f is strictly decreasing.]

The number s is called the **similarity dimension** of the fixed set of T.

6. Compute the similarity dimension of the Cantor set and of the Sierpinski gasket.