Solutions to Problem Set 2

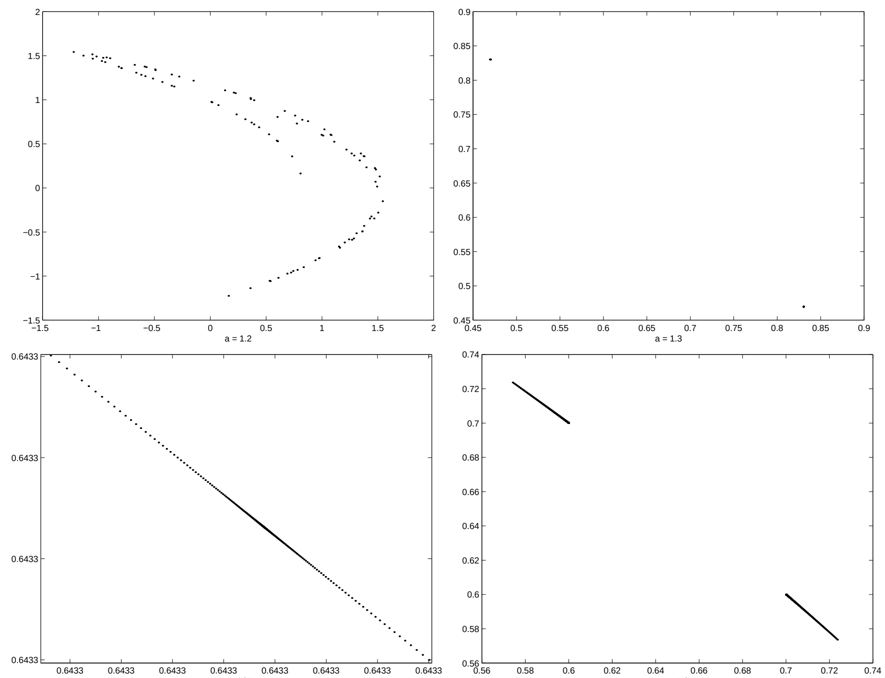
May 24, 2001

1. Here are a couple Matlab scripts I wrote:

```
henon.m

function h = henon(x,a)
% henon defines the henon map
h = x * [-x(1) 1; a(2) 0] + [a(1) 0];

henon_diagram.m

% henon_diagram.m.  Math 118, Spring 2001, PS 1, problem 1.
% The Henon map sends (x,y) to (a-x^2+by, x).
% This script asks for parameters a and b for the Henon map, a number
% of iterations k, and an initial point (x,y).
%
% It then calculates the 100th iterate of a random point,
% the next k iterations, and plots them.

a = input('Enter the parameters a,b for the Henon map as a row vector: ');
k = input('Enter the number of iterations k: ');
x = input('Enter the initial values x,y as a row vector: ');


A = zeros(k,2);


for n = 1:100
x = henon(x,a);
end

for m = 1:k
A(m,:) = x;
x = henon(x,a);
end
```

```
A = A';

plot(A(1,:),A(2,:),'r.');
```

2. Below are some printouts.



For values above $a = 1.2$, the diagram appears to indicate that there's a fixed point to which the iterates converge. But for values below $a = 1.3$, the diagram appears to indicate that the iterates converge to a period 2 orbit. So somewhere in between, it appears that there's a period-doubling bifurcation.

3. If $\mathbf{h}(x, y) = (x, y)$, then we have $x = a - x^2 + by$ and $y = x$, so we have $x = a - x^2 + bx$, or $x^2 + (1 - b)x - a = 0$. This equation has no real roots if $(1 - b)^2 < -4a$ or $a > -\frac{1}{4}(1 - b)^2$. It has one real root if $a = -\frac{1}{4}(1 - b)^2$ and two if $a > -\frac{1}{4}(1 - b)^2$.

4. If $\mathbf{h}(\mathbf{h}(x, y)) = (x, y)$, we have $(x, y) = (a - (a - x^2 + by)^2 + bx, a - x^2 + by)$. Thus $y = a - x^2 + by$, or $y = \frac{a - x^2}{1 - b}$. But looking at the first component of the period-2 equation, we have $x = a - (a - x^2 + by)^2 + bx$. Substituting for y, we get $x = a - (a - x^2 + \frac{b(a - x^2)}{1 - b})^2 + bx$, or $x = a - (a - x^2)^2(\frac{1}{1 - b})^2 + bx$. This can be rewritten as $x^4 - 2ax^2 + (1 - b)^3x + a(a - (1 - b)^2) = 0$. This factors as $(x^2 + (1-b)x - a)(x^2 - (1-b)x + (-a + (1-b)^2)) = 0$. The first factor is zero if and only if (x, y) is a fixed point. So we want the second factor to be zero. $x^2 - (1-b)x + (-a + (1-b)^2)) = 0$ has a solution if and only if $(1-b)^2 > -4a + 4(1-b)^2$, or if $a > \frac{3}{4}(1-b)^2$.

5. For $b = -.3$, the period two orbit appears when $a = \frac{3}{4}(1 - -.3)^2 = 1.2675$.

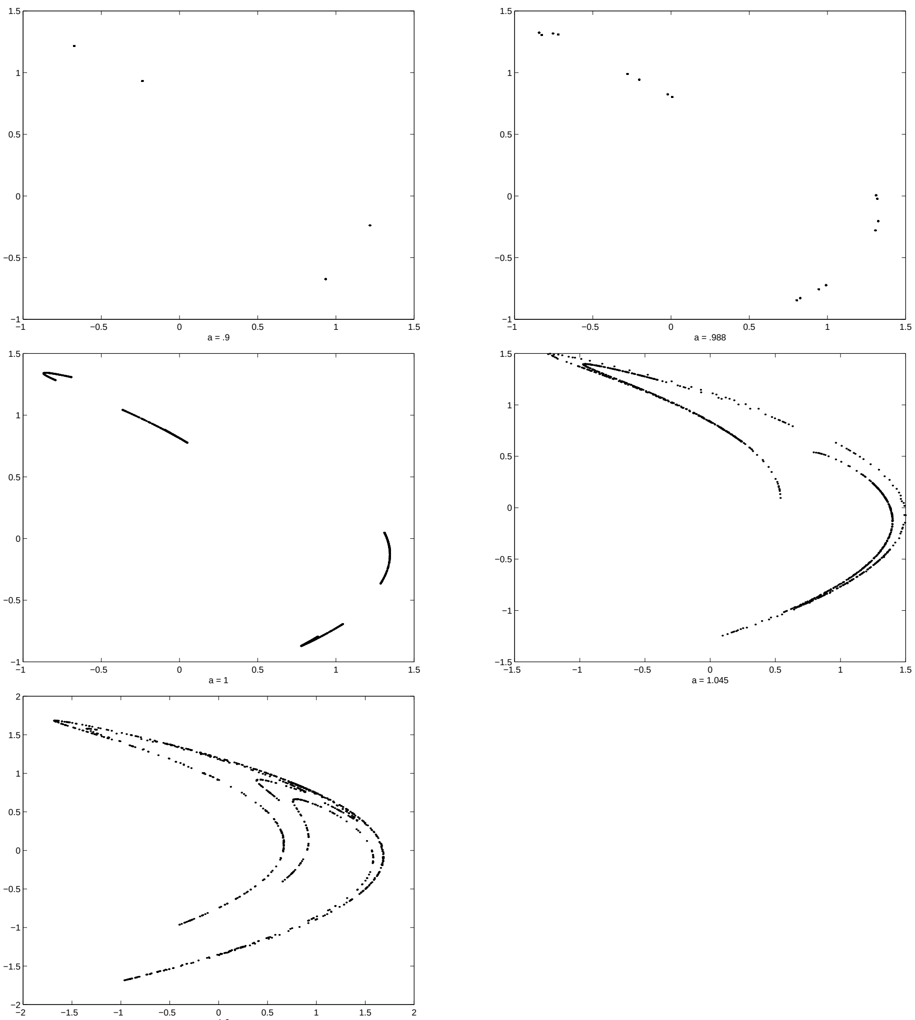
6. If at a particular point (x, y) the eigenvalues are complex, they are complex conjugates because the Jacobian is real. The product of the eigenvalues is the determinant of the Jacobian, which is $-b$. Hence each complex eigenvalue has magnitude $|b|^{\frac{1}{2}}$.

As a ranges from $-\frac{1}{4}(1-b)^2$ to $\frac{3}{4}(1-b)^2$, x_- decreases from $-\frac{1-b}{2}$ to $-\frac{3(1-b)}{2}$. $\lambda_+ = -x + \sqrt{x^2+b}$, so $lambda'_+(x) = -1 + \frac{x}{\sqrt{x^2+b}}$. Using the chain rule, $\frac{d}{da}\lambda_+(x_-(a)) = \lambda'_+(x_-(a))(x'_-(a))$. Now x'_- is negative for a in $(-\frac{1}{4}(1-b)^2, \frac{3}{4}(1-b)^2)$, and $lambda'_+$ is negative for x in $(-\frac{1-b}{2}, -\frac{3(1-b)}{2})$. Hence $\lambda_+(x_-(a))$ increases as a function of a, andd $lambda_+ = 1$ for $a = -\frac{1}{4}(1-b)^2$, $x_- = -\frac{1-b}{2}$. Thus $lambda_+ > 1$ for all a under consideration. Hence all the points $(x_-(a), y_-(a))$ are unstable.

Similarly, one can check that as a ranges from $-\frac{1}{4}(1-b)^2$ to $\frac{3}{4}(1-b)^2$, x_+ increases from $-\frac{1-b}{2}$ to $\frac{1-b}{2}$. Now $\lambda'_\pm = -1 \pm \frac{x}{\sqrt{x^2+b}}$. One can check that both λ'_+ and λ'_- are negative for x between $-\frac{1-b}{2}$ and $\frac{1-b}{2}$. Hence $\frac{d}{da}\lambda_\pm(x_+(a))$ are both negative. At $a = -\frac{1}{4}(1-b)^2$ the eigenvalues are $-b$ and 1. At $a = \frac{3}{4}(1-b)^2$ and $x_+ = \frac{1-b}{2}$, the eigenvalues are -1 and 0. Hence for all values a under consideration, both eigenvalues have magnitude less than 1. Hence the (x_+, y_+) are stable.

7. If (x_1, y_1) and (x_2, y_2) form a period-2 orbit then x_1 and x_2 are roots of $x^2 - (1-b)x + (-a + (1-b)^2) = 0$, so $x_1 + x_2 = 1 - b$. Since $\mathbf{h}(x_1, y_1) = (x_2, y_2)$ and $\mathbf{h}(x_2, y_2) = (x_1, y_1)$, we have $y_2 = x_1$ and $y_1 = x_2$, so $x_1 + y_1 = x_2 + y_2 = 1 - b$.

8. Below are the diagrams:


a = .9
a = .988
a = 1
a = 1.045
a=1.2