Math 118, Spring 2,001

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Contents

Chapter 2

Bifurcations

2.1 The logistic family.

In population biology one considers iteration of the "logistic function"

$$L_\mu(x) = \mu x(1-x). \tag{2.1}$$

Here $0 < \mu$ is a real parameter. The fixed points of L_μ are 0 and $1 - \frac{1}{\mu}$. Since $L'_\mu(x) = \mu - 2\mu x$,

$$L'_\mu(0) = \mu, \quad L'_\mu(1 - \frac{1}{\mu}) = 2 - \mu.$$

As x represents a proportion of a population, we are mainly interested only in $0 \leq x \leq 1$. The maximum of L_μ is always achieved at $x = \frac{1}{2}$, and the maximum value is $\frac{\mu}{4}$. So for $0 < \mu \leq 4$, L_μ maps $[0,1]$ into itself.

For $\mu > 4$, portions of $[0,1]$ are mapped into the range $x > 1$. A second operation of L_μ maps these points to the range $x < 0$ and then are swept off to $-\infty$ under successive applications of L_μ.

We now examine the behavior of L_μ more closely for varying ranges of μ.

2.1.1 $0 < \mu \leq 1$

For $0 < \mu < 1$, 0 is the only fixed point of L_μ on $[0,1]$ since the other fixed point is negative. On this range of μ, the point 0 is an attracting fixed point since $0 < L'_\mu(0) < 1$. Under iteration, all points of $[0,1]$ tend to 0 under the iteration. The population "dies out".

For $\mu = 1$ we have

$$L_1(x) = x(1-x) < x, \quad \forall x > 0.$$

Each successive application of L_1 to an $x \in (0,1]$ decreases its value. The limit of the successive iterates can not be positive since 0 is the only fixed point. So all points in $(0,1]$ tend to 0 under iteration, but ever so slowly, since $L'_1(0) = 1$.

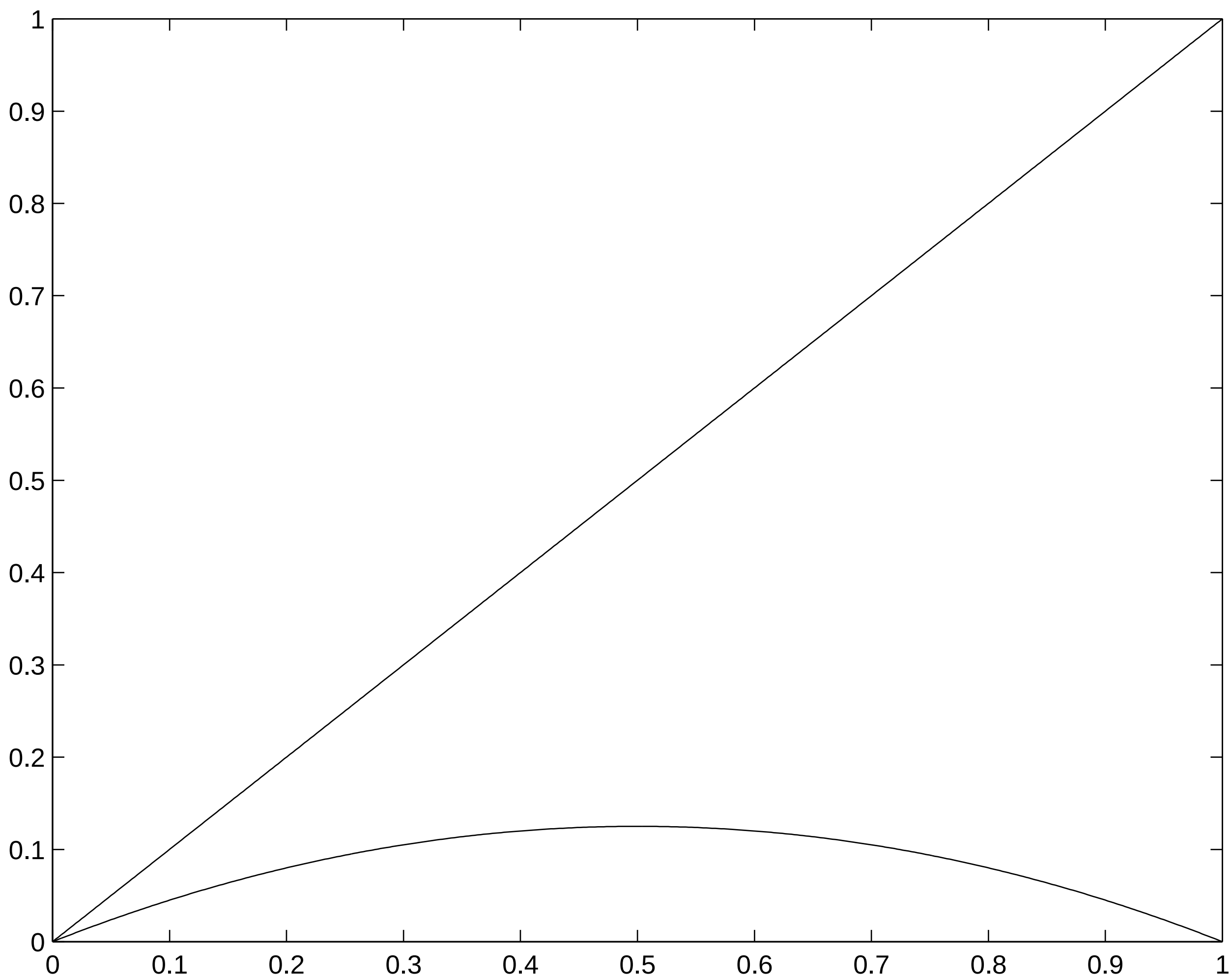

Figure 2.1: $\mu = .5$

In fact, for $x < 0$, the iterates drift off to more negative values and then tend to $-\infty$.

For all $\mu > 1$, the fixed point, 0, is repelling, and the unique other fixed point, $1 - \frac{1}{\mu}$, lies in $[0, 1]$. For $1 < \mu < 3$ we have

$$|L'_\mu(1 - \frac{1}{\mu})| = |2 - \mu| < 1,$$

so the non-zero fixed point is attractive.

We will see that the basin of attraction of $1 - \frac{1}{\mu}$ is the entire open interval $(0, 1)$, but the behavior is slightly different for the two domains, $1 < \mu \leq 2$ and $2 < \mu < 3$:

In the first of these ranges there is a steady approach toward the fixed point from one side or the other; in the second, the iterates bounce back and forth from one side to the other as they converge in towards the fixed point. The graphical iteration spirals in. Here are the details:

2.1.2 $1 < \mu \leq 2$.

For $1 < \mu < 2$ the non-zero fixed point lies between 0 and $\frac{1}{2}$ and the derivative at this fixed point is $2 - \mu$ and so lies between 1 and 0.

Suppose that x lies between 0 and $1 - \frac{1}{\mu}$. For this range of x we have

$$\frac{1}{\mu} < 1 - x$$

so, multiplying by μx we get

$$x < \mu x(1 - x) = L_\mu(x).$$

Thus the iterates steadily increase toward $1 - \frac{1}{\mu}$, eventually converging geometrically with a rate close to $2 - \mu$. If

$$1 - \frac{1}{\mu} < x$$

then

$$L_\mu(x) < x.$$

If, in addition,

$$x \leq \frac{1}{\mu}$$

then

$$L_\mu(x) \geq 1 - \frac{1}{\mu}.$$

To see this observe that the function L_μ has only one critical point, and that is a maximum. Since $L_\mu(1 - \frac{1}{\mu}) = L_\mu(\frac{1}{\mu}) = 1 - \frac{1}{\mu}$, we conclude that the minimum value is achieved at the end points of the interval $[1 - \frac{1}{\mu}, \frac{1}{\mu}]$.

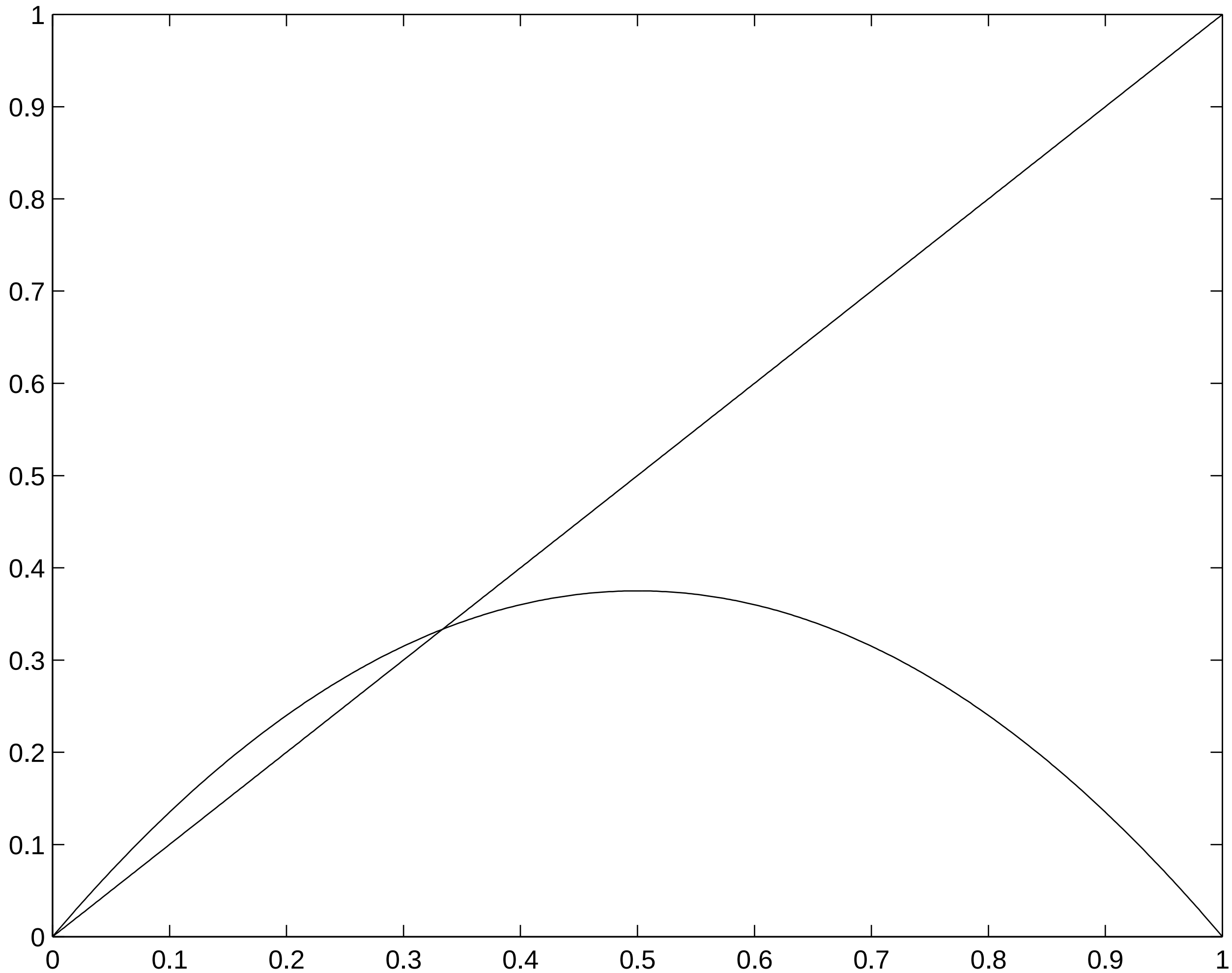

Figure 2.2: $\mu = 1.5$

Finally, for

$$\frac{1}{\mu} < x \le 1, \quad L_\mu(x) < 1 - \frac{1}{\mu}.$$

So on the range $1 < \mu < 2$ the behavior of L_μ is as follows: All points $0 < x < 1 - \frac{1}{\mu}$ steadily increase toward the fixed point, $1 - \frac{1}{\mu}$. All points satisfying $1 - \frac{1}{\mu} < x < \frac{1}{\mu}$ steadily decrease toward the fixed point. The point $\frac{1}{\mu}$ satisfies $L_\mu(\frac{1}{\mu}) = 1 - \frac{1}{\mu}$ and so lands on the non-zero fixed point after one application. The points satisfying $\frac{1}{\mu} < x < 1$ get mapped by L_μ into the interval $0 < x < 1 - \frac{1}{\mu}$, In other words, they overshoot the mark, but then steadily increase towards the non-zero fixed point. Of course $L_\mu(1) = 0$ which is always true.

When $\mu = 2$, the points $\frac{1}{\mu}$ and $1 - \frac{1}{\mu}$ coincide and equal $\frac{1}{2}$ with $L_2'(\frac{1}{2}) = 0$. There is no "steadily decreasing" region, and the fixed point, $\frac{1}{2}$ is superattractive - the iterates zoom into the fixed point faster than any geometrical rate.

2.1.3 $2 < \mu < 3$.

Here the fixed point $1 - \frac{1}{\mu} > \frac{1}{2}$ while $\frac{1}{\mu} < \frac{1}{2}$. The derivative at this fixed point is negative:

$$L_\mu'(1 - \frac{1}{\mu}) = 2 - \mu < 0.$$

So the fixed point $1 - \frac{1}{\mu}$ is an attractor, but as the iterates converge to the fixed points, they oscillate about it, alternating from one side to the other. The entire interval $(0, 1)$ is in the basin of attraction of the fixed point. To see this, we may argue as follows:

The graph of L_μ lies entirely above the line $y = x$ on the interval $(0, 1 - \frac{1}{\mu}]$. In particular, it lies above the line $y = x$ on the subinterval $[\frac{1}{\mu}, 1 - \frac{1}{\mu}]$ and takes its maximum at $\frac{1}{2}$. So $\frac{\mu}{4} = L_\mu(\frac{1}{2}) > L_\mu(1 - \frac{1}{\mu}) = 1 - \frac{1}{\mu}$. Hence L_μ maps the interval $[\frac{1}{\mu}, 1 - \frac{1}{\mu}]$ onto the interval $[1 - \frac{1}{\mu}, \frac{\mu}{4}]$. The map L_μ is decreasing to the right of $\frac{1}{2}$, so it is certainly decreasing to the right of $1 - \frac{1}{\mu}$. Hence it maps the interval $[1 - \frac{1}{\mu}, \frac{\mu}{4}]$ into an interval whose right hand end point is $1 - \frac{1}{\mu}$ and whose left hand end point is $L_\mu(\frac{\mu}{4})$. We claim that

$$L_\mu(\frac{\mu}{4}) > \frac{1}{2}.$$

This amounts to showing that

$$\frac{\mu^2(4 - \mu)}{16} > \frac{1}{2}$$

or that

$$\mu^2(4 - \mu) > 8.$$

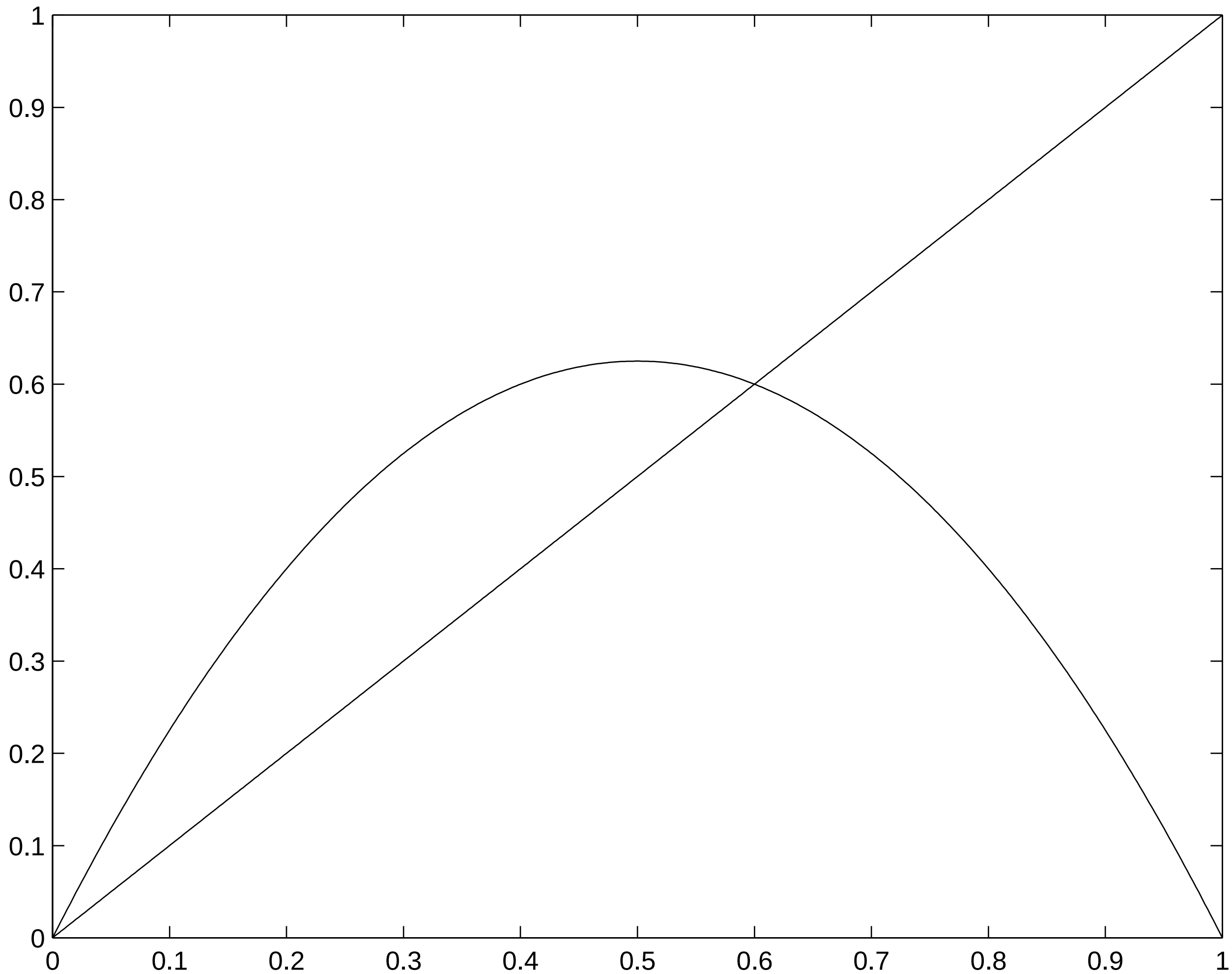

Figure 2.3: $\mu = 2.5,\ \frac{1}{\mu} = .4,\ 1 - \frac{1}{\mu} = .6$

Now the critical points of $\mu^2(4-\mu)$ are 0 and $\frac{8}{3}$ and the second derivative at $\frac{8}{3}$ is negative, so it is a local maximum. So we need only check the values of $\mu^2(4-\mu)$ at the end points, 2 and 3, of the range of μ we are considering, where the values are 8 and 9.

The image of $[\frac{1}{\mu}, 1-\frac{1}{\mu}]$ is the same as the image of $[\frac{1}{2}, 1-\frac{1}{\mu}]$ and is $[1-\frac{1}{\mu}, \frac{\mu}{4}]$. The image of this interval is the interval $[L_\mu(\frac{\mu}{4}), 1-\frac{1}{\mu}]$, with $\frac{1}{2} < L_\mu(\frac{\mu}{4})$. If we apply L_μ to this interval, we get an interval to the right of $1-\frac{1}{\mu}$ with right end point $L_\mu^2(\frac{\mu}{4}) < L_\mu(\frac{1}{2}) = \frac{\mu}{4}$. The image of the interval $[1-\frac{1}{\mu}, L_\mu^2(\frac{\mu}{4})]$ must be strictly contained in the image of the interval $[1-\frac{1}{\mu}, \frac{\mu}{4}]$, and hence we conclude that

$$L_\mu^3(\frac{\mu}{4}) > L_\mu(\frac{\mu}{4}).$$

Continuing in this way we see that under even powers, the image of $[\frac{1}{2}, 1-\frac{1}{\mu}]$ is a sequence of nested intervals whose right hand end point is $1-\frac{1}{\mu}$ and whose left hand end points are

$$\frac{1}{2} < L_\mu(\frac{\mu}{4}) < L_\mu^3(\frac{\mu}{4}) < \cdots .$$

We claim that this sequence of points converges to the fixed point, $1-\frac{1}{\mu}$. If not, it would have to converge to a fixed point of L_μ^2 different from 0 and $1-\frac{1}{\mu}$. We shall show that there are no such points. Indeed, a fixed point of L_μ^2 is a zero of

$$L_\mu^2(x) - x = \mu L_\mu(x)(1-L_\mu(x)) = \mu[\mu x(1-x)][1-\mu x(1-x)] - x.$$

We know in advance two roots of this quartic polynomial, namely the fixed points of L_μ,which are 0 and $1-\frac{1}{\mu}$. So we know that the quartic polynomial factors into a quadratic polynomial times $\mu x(x-1+\frac{1}{\mu})$. A direct check shows that this quadratic polynomial is

$$-\mu^2 x^2 + (\mu^2+\mu)x - \mu - 1. \tag{2.2}$$

The b^2-4ac for this quadratic function is

$$\mu^2(\mu^2-2\mu-3) = \mu^2(\mu+1)(\mu-3)$$

which is negative for $-1 < \mu < 3$ and so (2.2) has no real roots.

We thus conclude that the iterates of any point in $(\frac{1}{\mu}, \frac{\mu}{4}]$ oscillate about the fixed point, $1-\frac{1}{\mu}$ and converge in towards it, eventually with the geometric rate of convergence a bit less than $\mu-2$. The graph of L_μ is strictly above the line $y=x$ on the interval $(0, \frac{1}{\mu}]$ and hence the iterates of L_μ are strictly increasing so long as they remain in this interval. Furthermore they can't stay there, for this would imply the existence of a fixed point in the interval and we know that there is none. Thus they eventually get mapped into the interval $[\frac{1}{\mu}, 1-\frac{1}{\mu}]$ and the oscillatory convergence takes over. Finally, since L_μ is decreasing on

$[1-\frac{1}{\mu}, 1]$, any point in $[1-\frac{1}{\mu}, 1)$ is mapped into $(0, 1-\frac{1}{\mu}]$ and so converges to the non-zero fixed point.

In short, every point in $(0,1)$ is in the basin of attraction of the non-zero fixed point and (except for the points $\frac{1}{\mu}$ and the fixed point itself) eventually converge toward it in a "spiral" fashion.

2.1.4 $\mu = 3$.

Much of the analysis of the preceding case applies here. The differences are: the quadratic equation (2.2) now has a (double) root. But this root is $\frac{2}{3} = 1 - \frac{1}{\mu}$. So there is still no point of period two other than the fixed points. The iterates continue to spiral in, but now ever so slowly since $L'_\mu(\frac{2}{3}) = -1$.

For $\mu > 3$ we have

$$L'_\mu(1 - \frac{1}{\mu}) = 2 - \mu < -1$$

so both fixed points, 0 and $1 - \frac{1}{\mu}$ are repelling. But now (2.2) has two real roots which are

$$p_{2\pm} = \frac{1}{2} + \frac{1}{2\mu} \pm \frac{1}{2\mu}\sqrt{(\mu+1)(\mu-3)}.$$

Both roots lie in $(0,1)$ and give a period two cycle for L_μ. The derivative of L_μ^2 at these periodic points is given by

$$\begin{aligned}
(L_\mu^2)'(p_{2\pm}) &= L'_\mu(p_{2+})L'_\mu(p_{2-}) \\
&= (\mu - 2\mu p_{2+})(\mu - 2\mu p_{2-}) \\
&= \mu^2 - 2\mu^2(p_{2+} + p_{2-}) + 4\mu^2 p_{2+}p_{2-} \\
&= \mu^2 - 2\mu^2(1 + \frac{1}{\mu}) + 4\mu^2 \times \frac{1}{\mu^2}(\mu+1) \\
&= -\mu^2 + 2\mu + 4.
\end{aligned}$$

This last expression equals 1 when $\mu = 3$ as we already know. It decreases as μ increases reaching the value -1 when $\mu = 1 + \sqrt{6}$.

2.1.5 $3 < \mu < 1 + \sqrt{6}$.

In this range the fixed points are repelling and both period two points are attracting. There will be points whose images end up, after a finite number of iterations, on the non-zero fixed point. All other points in $(0,1)$ are attracted to the period two cycle. We omit the proof.

Notice also that there is a unique value of μ in this range where

$$p_{2+}(\mu) = \frac{1}{2}.$$

Indeed, looking at the formula for p_{2+} we see that this amounts to the condition that $\sqrt{(\mu+1)(\mu-3)} = 1$ or

$$\mu^2 - 2\mu - 4 = 0.$$

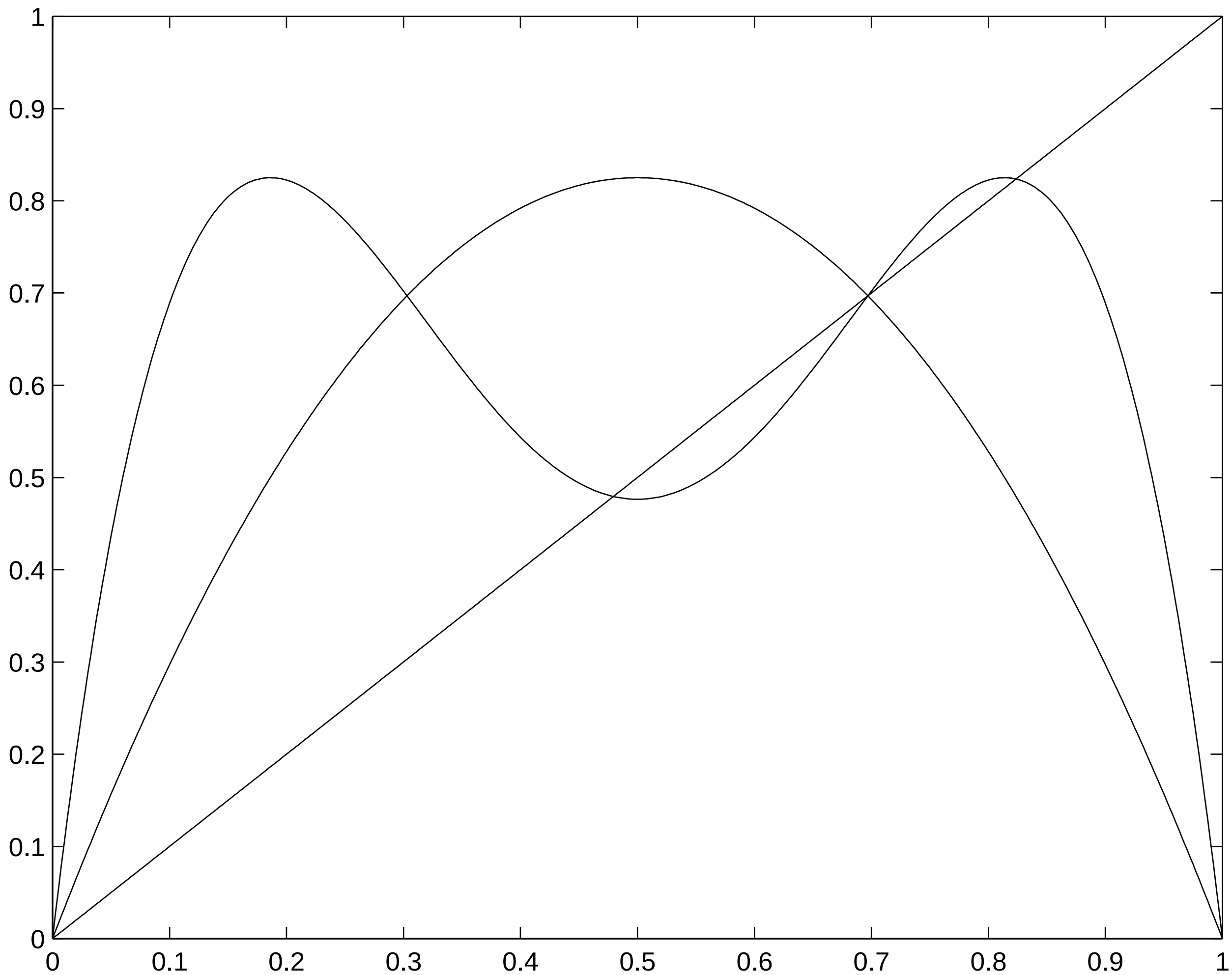

Figure 2.4: $\mu = 3.3$, graphs of $y = x, y = L_\mu(x),\ y = L_\mu^2(x)$.

The positive solution to this equation is given by $\mu = s_2$ where

$$s_2 = 1 + \sqrt{5}.$$

At s_2, the period two points are superattracting, since one of them coincides with $\frac{1}{2}$ which is the maximum of L_{s_2}.

2.1.6 $3.449499... < \mu < 3.569946....$

Once μ passes $1 + \sqrt{6} = 3.449499...$ the points of period two become unstable and (stable) points of period four appear. Initially these are stable, but as μ increases they become unstable (at the value $\mu = 3.544090...$) and bifurcate into period eight points, initially stable.

2.1.7 Reprise.

The total scenario so far, as μ increases from 0 to about 3.55, is as follows: For $\mu < b_1 := 1$, there is no non-zero fixed point. Past the first bifurcation point, $b_1 = 1$, the non-zero fixed point has appeared close to zero. When μ reaches the first superattractive value , $s_1 := 2$, the fixed point is at .5 and is superattractive. As μ increases, the fixed point continues to move to the right. Just after the second bifurcation point, $b_2 := 3$, the fixed point has become unstable and two stable points of period two appear, one to the right and one to the left of .5. The leftmost period two point moves to the right as we increase μ, and at $\mu = s_2 := 1 + \sqrt{5} = 3.23606797...$ the point .5 is a period two point, and so the period two points are superattractive. When μ passes the second bifurcation value $b_2 = 1 + \sqrt{6} = 3.449..$ the period two points have become repelling and attracting period four points appear.

In fact, this scenario continues. The period 2^{n-1} points appear at bifurcation values b_n. They are initially attracting, and become superattracting at $s_n > b_n$ and become unstable past the next bifurcation value $b_{n+1} > s_n$ when the period 2^n points appear. The (numerically computed) bifurcation points and superstable points are tabulated as

n	b_n	s_n
1	1.000000	2.000000
2	3.000000	3.236068
3	3.449499	3.498562
4	3.544090	3.554641
5	3.564407	3.566667
6	3.568759	3.569244
7	3.569692	3.569793
8	3.569891	3.569913
9	3.569934	3.569946
∞	3.569946	3.569946

The values of the b_n are obtained by numerical experiment. We shall describe a method for computing the s_n using Newton's method. We should point out that this is still just the beginning of the story. For example, an attractive period three cycle appears at about 3.83. We shall come back to all of these points, but first discuss theoretical problems associated to bifurcations.

2.2 Local bifurcations.

We will be studying the iteration (in x) of a function, F, of two real variables x and μ . We will need to make various hypothesis concerning the differentiability of F. We will always assume usually it is at least C^2 (has continuous partial derivatives up to the second order). We may also need C^3 in which case we explicitly state this hypothesis. We write

$$F_\mu(x) = F(x, \mu)$$

and are interested in the change of behavior of F_μ as μ varies.

Before embarking on the study of bifurcations let us observe that if p is a fixed point of F_μ and $F_\mu'(p) \neq 1$, then for ν close to μ, the transformation F_ν has a unique fixed point close to p. Indeed, the implicit function theorem applies to the function

$$P(x, \nu) := F(x, \nu) - x$$

since

$$\frac{\partial P}{\partial x}(p, \mu) \neq 0$$

by hypothesis. We conclude that there is a curve of fixed points $x(\nu)$ with $x(\mu) = p$.

2.2.1 The fold.

The first type of bifurcation we study is the *fold bifurcation* where there is no (local) fixed point on one side of the bifurcation value, b, where a fixed point p appears at $\mu = b$ with $F_\mu'(p) = 1$, and at the other side of b the map F_μ has two fixed points, one attracting and the other repelling.

As an example consider the quadratic family

$$Q(x, \mu) = Q_\mu(x) := x^2 + \mu.$$

Fixed points must be solutions of the quadratic equation

$$x^2 - x + \mu = 0,$$

whose roots are

$$p_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\mu}.$$

For

$$\mu > b = \frac{1}{4}$$

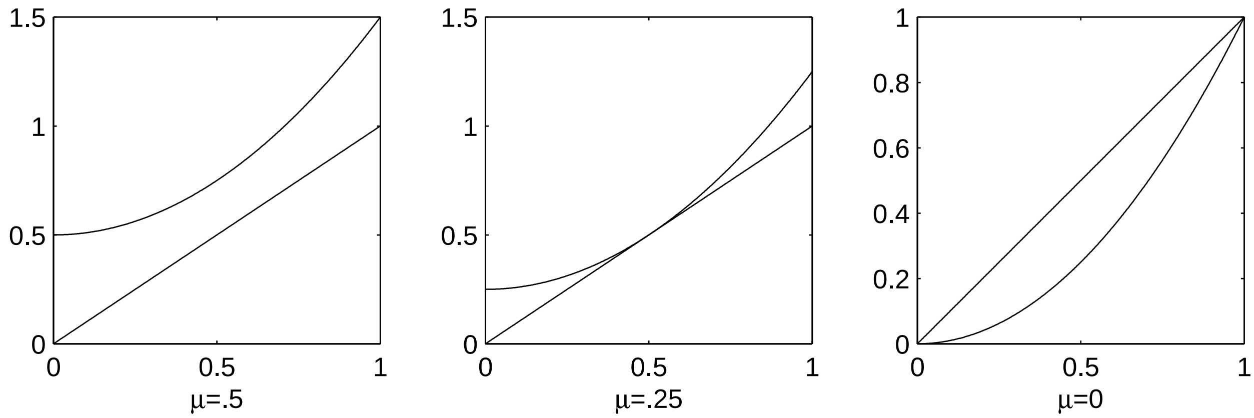

Figure 2.5: $y = x^2 + \mu$ for $\mu = .5,\ .25$ and 0.

these roots are not real. The parabola $x^2 + \mu$ lies entirely above the line $y = x$ and there are no fixed points.

At $\mu = \frac{1}{4}$ the parabola just touches the line $y = x$ at the point $(\frac{1}{2}, \frac{1}{2})$ and so

$$p = \frac{1}{2}$$

is a fixed point, with $Q'_\mu(p) = 2p = 1$.

For $\mu < \frac{1}{4}$ the points $p_\pm$ are fixed points, with $Q'_\mu(p_+) > 1$ so it is repelling, and $Q'_\mu(p_-) < 1$. We will have $Q'_\mu(p_-) > -1$ so long as $\mu > -\frac{3}{4}$, so on the range $-\frac{3}{4} < \mu < \frac{1}{4}$ we have two fixed points, one repelling and one attracting.

We will now discuss the general phenomenon. In order not to clutter up the notation, we assume that coordinates have been chosen so that $b = 0$ and $p = 0$. So we make the standing assumption that $p = 0$ is a fixed point at $\mu = 0$, i.e. that

$$F(0,0) = 0.$$

Proposition 2.2.1 (Fold bifurcation). *Suppose that at the point* $(0,0)$ *we have*

$$\text{(a)}\ \frac{\partial F}{\partial x}(0,0) = 1, \qquad \text{(b)}\ \frac{\partial^2 F}{\partial x^2}(0,0) > 0, \qquad \text{(c)}\ \frac{\partial F}{\partial \mu}(0,0) > 0.$$

Then there are non-empty intervals $(\mu_1, 0)$ *and* $(0, \mu_2)$ *and* $\epsilon > 0$ *so that*

(i) *If* $\mu \in (\mu_1, 0)$ *then* F_μ *has two fixed points in* $(-\epsilon, \epsilon)$.
One is attracting and the other repelling.

(ii) *If* $\mu \in (0, \mu_2)$ *then* F_μ *has no fixed points in* $(-\epsilon, \epsilon)$.

Proof. All the proofs in this section will be applications of the implicit function theorem. For our current proposition, set

$$P(x, \mu) := F(x, \mu) - x.$$

Then by our standing hypothesis we have

$$P(0, 0) = 0$$

and condition (c) gives

$$\frac{\partial P}{\partial \mu}(0, 0) > 0.$$

The implicit function theorem gives a unique function $\mu(x)$ with $\mu(0) = 0$ and

$$P(x, \mu(x)) \equiv 0.$$

The formula for the derivative in the implicit function theorem gives

$$\mu'(x) = -\frac{\partial P/\partial x}{\partial P/\partial \mu}$$

which vanishes at the origin by assumption (a). We then may compute the second derivative, μ'', via the chain rule; using the fact that $\mu'(0) = 0$ we obtain

$$\mu''(0) = -\frac{\partial^2 P/\partial x^2}{\partial P/\partial \mu}(0, 0).$$

This is negative by assumptions (b) and (c). In other words,

$$\mu'(0) = 0, \text{ and } \mu''(0) < 0$$

so $\mu(x)$ has a maximum at $x = 0$, and this maximum value is 0. In the (x, μ) plane, the graph of $\mu(x)$ looks locally like a parabola pointing in the lower half plane with its apex at the origin. If we rotate this picture clockwise by ninety degrees, this says that there are no points on this curve sitting over positive μ values, i.e. no fixed points for positive μ, and two fixed points for $\mu < 0$.

Now consider the function $\frac{\partial F}{\partial x}(x, \mu(x))$. The derivative of this function with respect to x is

$$\frac{\partial^2 F}{\partial x^2}(x, \mu(x)) + \frac{\partial^2 F}{\partial x \partial \mu}(x, \mu(x))\mu'(x).$$

By assumption (b) and $\mu'(0) = 0$, this expression is positive at $x = 0$ and so $\frac{\partial F}{\partial x}(x, \mu(x))$ is an increasing function in a neighborhood of the origin while $\frac{\partial F}{\partial x}(0, 0) = 1$. But this says that

$$F'_\mu(x) < 1$$

on the lower fixed point and

$$F'_\mu(x) > 1$$

at the upper fixed point, completing the proof of the proposition. We should point out that changing the sign in (b) or (c) interchanges the role of the two intervals.

2.2.2 Period doubling.

We now turn to the *period doubling bifurcation.* This is what happens when we pass through a bifurcation value with

$$\frac{\partial F}{\partial x}(0,0) = -1. \tag{2.3}$$

We saw examples in the preceding section.

To visualize the phenomenon we plot the function $L_\mu^{\circ 2}$ for the values $\mu = 2.9$ and $\mu = 3.3$ in Figure 2.6. For $\mu = 2.9$ the curve crosses the diagonal at a single point, which is in fact a fixed point of L_μ and hence of $L_\mu^{\circ 2}$. This fixed point is stable. For $\mu = 3.3$ there are three crossings. The fixed point of L_μ has derivative smaller than -1, and hence the corresponding fixed point of $L_\mu^{\circ 2}$ has derivative greater than one. The two other crossings correspond to the stable period two orbit.

We now turn to the general theory: Notice that the partial derivative of $F(x,\mu) - x$ with respect to x is -2 at the origin. In particular it does not vanish, so we can now solve for x as a function of μ; there is a unique branch of fixed points, $x(\mu)$, passing through the origin. Let $\lambda(\mu)$ denote the derivative of F_μ with respect to x at the fixed point, $x(\mu)$, i.e. define

$$\lambda(\mu) := \frac{\partial F}{\partial x}(x(\mu),\mu).$$

As notation, let us set

$$F_\mu^{\circ 2} := F_\mu \circ F_\mu$$

and define

$$F^{\circ 2}(x,\mu) := F_\mu^{\circ 2}(x).$$

Notice that

$$(F_\mu^{\circ 2})'(x) = F_\mu'(F_\mu(x))F_\mu'(x)$$

by the chain rule so

$$(F_0^{\circ 2})'(0) = (F_0'(0))^2 = 1.$$

Hence

$$(F_\mu^{\circ 2})''(x) = F_\mu''(F_\mu(x))F_\mu'(x)^2 + F_\mu'(F_\mu(x))F_\mu''(x) \tag{2.4}$$

which vanishes at $x = 0, \mu = 0$. In other words,

$$\frac{\partial^2 F^{\circ 2}}{\partial x^2}(0,0) = 0. \tag{2.5}$$

Let us absorb the import of this equation. One might think that if we set $G_\mu = F_\mu^{\circ 2}$, then $G_\mu'(0) = 1$, so all we need to do is apply Proposition 1 to G_μ. But (2.5) shows that the key condition (b) of Proposition 1 is violated, and hence we must make some alternative hypotheses. The hypotheses that we will

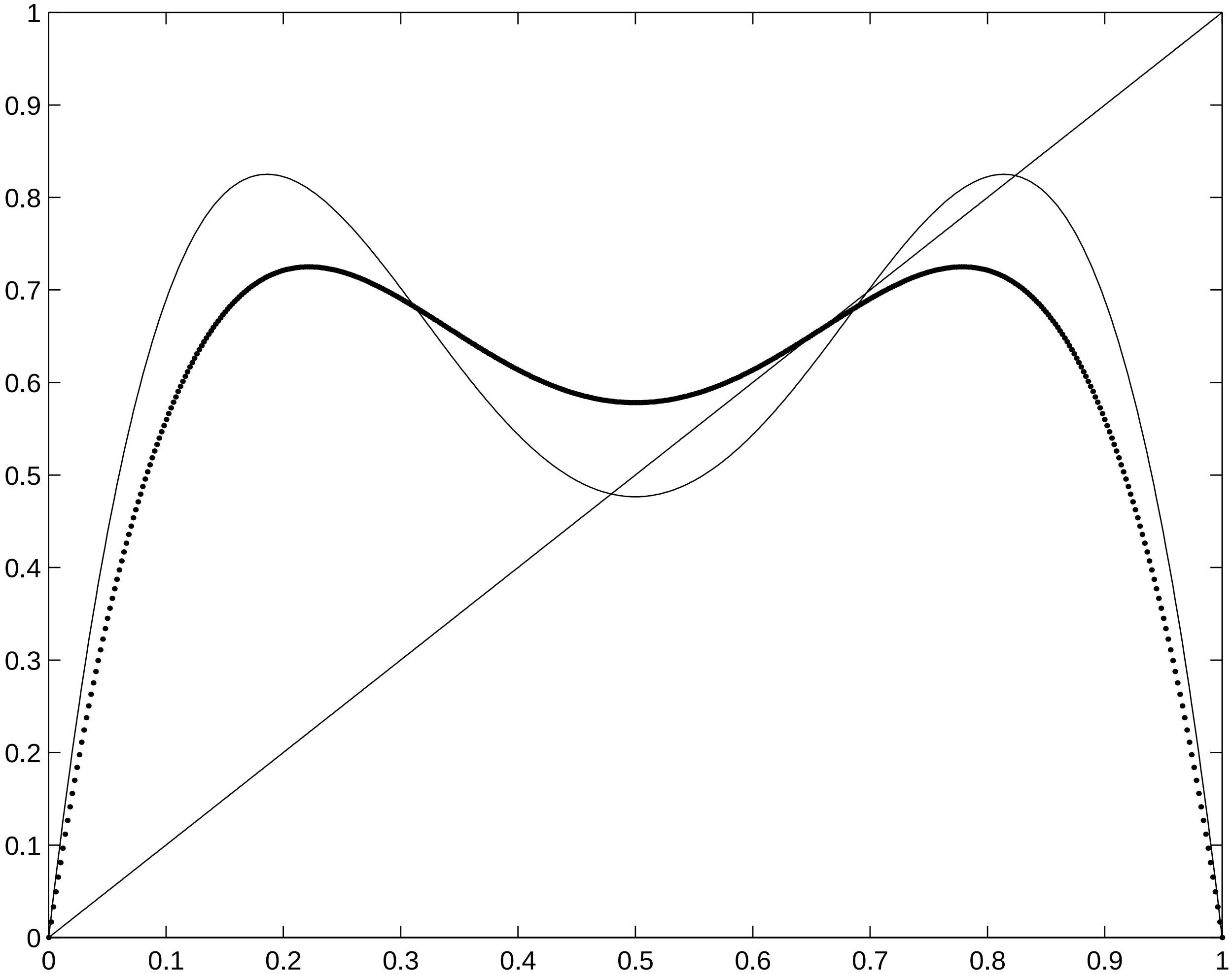

Figure 2.6: Plots of $L_\mu^{\circ 2}$ for $\mu = 2.9$ (dotted curve) and $\mu = 3.3$.

make will involve the second and the third partial derivatives of F, and also that $\lambda(\mu)$ really passes through -1, i.e. $\frac{d\lambda}{d\mu}(0) \neq 0$.

To understand the hypothesis about involving the partial derivatives of F, let us differentiate (2.4) once more with respect to x to obtain

$$(F_\mu^{\circ 2})'''(x) =$$

$$F_\mu'''(F_\mu(x))F_\mu'(x)^3 + 2F_\mu''(F_\mu(x))F_\mu''(x)F_\mu'(x) + F_\mu''(F_\mu(x))F_\mu'(x)F_\mu''(x) + F_\mu'(F_\mu(x))F_\mu'''(x).$$

At $(x,\mu) = (0,0)$ this simplifies to

$$-\left[2\frac{\partial^3 F}{\partial x^3}(0,0) + 3\frac{\partial^2 F}{\partial x^2}(0,0)\right]. \tag{2.6}$$

Proposition 2.2.2 (Period doubling bifurcation). *Suppose that F is C^3, that*

$$\text{(d)} F_0'(0) = -1 \quad \text{(e)} \quad \frac{d\lambda}{d\mu}(0) > 0, \quad \textit{and} \quad \text{(f)} \quad 2\frac{\partial^3 F}{\partial x^3}(0,0) + 3\frac{\partial^2 F}{\partial x^2}(0,0) > 0.$$

Then there are non-empty intervals $(\mu_1, 0)$ and $(0, \mu_2)$ and $\epsilon > 0$ so that

(i) *If $\mu \in (\mu_1, 0)$ then F_μ has one repelling fixed point and one attracting orbit of period two in $(-\epsilon, \epsilon)$*

(ii) *If $\mu \in (0, \mu_2)$ then $F_\mu^{\circ 2}$ has a single fixed point in $(-\epsilon, \epsilon)$ which is in fact an attracting fixed point of F_μ.*

The statement of the theorem is summarized in Figure 2.7:

Proof. Let

$$H(x,\mu) := F^{\circ 2}(x,\mu) - x.$$

Then by the remarks before the proposition, H vanishes at the origin together with its first two partial derivatives with respect to x. The expression (2.6) (which used condition (d)) together with conditions (f) gives

$$\frac{\partial^3 H}{\partial x^3}(0,0) < 0.$$

One of the zeros of H corresponds to the fixed point, let us factor this out: Define $P(x,\mu)$ by

$$H(x,\mu) = (x - x(\mu))P(x,\mu). \tag{2.7}$$

Then

$$\begin{aligned}
\frac{\partial H}{\partial x} &= P + (x-\mu)\frac{\partial P}{\partial x} \\
\frac{\partial^2 H}{\partial x^2} &= 2\frac{\partial P}{\partial x} + (x - x(\mu))\frac{\partial^2 P}{\partial x^2} \\
\frac{\partial^3 H}{\partial x^3} &= 3\frac{\partial^2 P}{\partial x^2} + (x - x(\mu))\frac{\partial^3 H}{\partial x^3}.
\end{aligned}$$

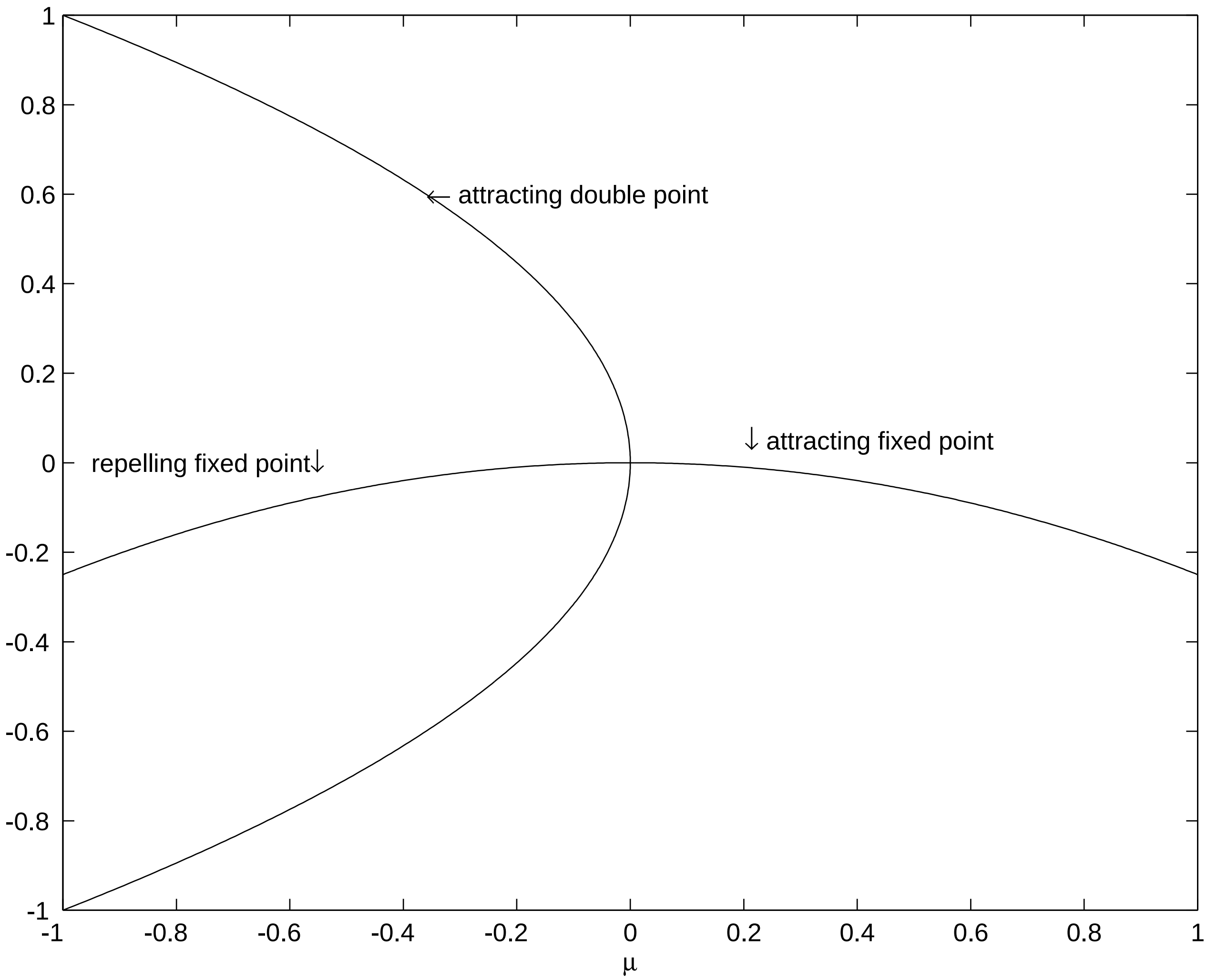

Figure 2.7: Period doubling bifurcation.

So P vanishes at the origin together with its first partial derivative with respect to x, while

$$\frac{\partial^3 H}{\partial x^3}(0,0) = 3\frac{\partial^2 P}{\partial x^2}(0,0)$$

so

$$\frac{\partial^2 P}{\partial x^2}(0,0) < 0. \tag{2.8}$$

We claim that

$$\frac{\partial P}{\partial \mu}(0,0) < 0, \tag{2.9}$$

so that we can apply the implicit function theorem to $P(x,\mu) = 0$ to solve for μ as a function of x. This will allow us to determine the fixed points of $F_\mu^{\circ 2}$ which are *not* fixed points of F_μ, i.e. the points of period two. To prove (2.9) we compute $\frac{\partial H}{\partial x}$ both from its definition $H(x,\mu) = F^{\circ 2}(x,\mu) - x$ and from (2.7) to obtain

$$\begin{aligned} \frac{\partial H}{\partial x} &= \frac{\partial F}{\partial x}(F(x,\mu),\mu)\frac{\partial F}{\partial x}(x,\mu) - 1 \\ &= P(x,\mu) + (x - x(\mu))\frac{\partial P}{\partial x}(x,\mu). \end{aligned}$$

Recall that $x(\mu)$ is the fixed point of F_μ and that $\lambda(\mu) = \frac{\partial F}{\partial x}(x(\mu),\mu)$. So substituting $x = x(\mu)$ into the preceding equation gives

$$\lambda(\mu)^2 - 1 = P(x,\mu).$$

Differentiating with respect to μ and setting $\mu = 0$ gives

$$\frac{\partial P}{\partial \mu}(0,0) = 2\lambda(0)\lambda'(0) = -2\lambda'(0)$$

which is < 0 by (e).

By the implicit function theorem, (2.9) implies that there is a C^2 function $\nu(x)$ defined near zero as the unique solution of $P(x,\nu(x)) \equiv 0$. Recall that P and its first derivative with respect to x vanish at $(0,0)$. We now repeat the arguments of the last subsection: We have

$$\nu'(x) = -\frac{\partial P/\partial x}{\partial P/\partial \mu}$$

so

$$\nu'(0) = 0$$

and

$$\nu''(0) = -\frac{\partial^2 P/\partial x^2}{\partial P/\partial x}(0,0) < 0$$

since this time both numerator and denominator are negative. So the curve ν has the same form as in the proof of the preceding proposition. This establishes the existence of the (strictly) period two points for $\mu < 0$ and their absence for $\mu > 0$.

We now turn to the question of the stability of the fixed points and the period two points. Condition (e) implies that $\lambda(\mu) < -1$ for $\mu < 0$ and $\lambda(\mu) > -1$ for $\mu > 0$ so the fixed point is repelling to the left and attracting to the right of the origin. As for the period two points, we wish to show that

$$\frac{\partial F^{\circ 2}}{\partial x}(x, \nu(x)) < 1$$

for $x < 0$. But (2.5) and $\nu'(0) = 0$ imply that imply that 0 is a critical point for this function, and the value at this critical point is $\lambda(0)^2 = 1$. To complete the proof we must show that this critical point is a local maximum. So we must compute the second derivative at the origin. Calling this function ϕ we have

$$\begin{aligned}
\phi(x) &:= \frac{\partial F^{\circ 2}}{\partial x}(x, \nu(x)) \\
\phi'(x) &= \frac{\partial^2 F^{\circ 2}}{\partial x^2}(x, \nu(x)) + \frac{\partial^2 F^{\circ 2}}{\partial x \partial \mu}(x, \nu(x))\nu'(x) \\
\phi''(x) &= \frac{\partial^3 F^{\circ 2}}{\partial x^3}(x, \nu(x)) + 2\frac{\partial^3 F^{\circ 2}}{\partial x^2 \partial \mu}(x, \nu(x)\nu'(x)) + \frac{\partial^3 F^{\circ 2}}{\partial x \partial \mu^2}(x, \nu(x))(\nu'(x))^2 + \\
&\quad \frac{\partial^2 F^{\circ 2}}{\partial x \partial \mu}(x, \nu(x))\nu''(x).
\end{aligned}$$

The middle two terms vanish at 0 since $\nu'(0) = 0$. The last term becomes

$$\frac{d\lambda}{d\mu}(0)\nu''(0) < 0$$

by condition (e) and the fact that $\nu''(0) < 0$. In computing the third partial derivative with respect to x of $F^{\circ 2}$ by the chain rule and by Leibniz's rule, all terms involving the second partial derivative vanish at $(0, 0)$ by (2.5) and we are left with

$$2\frac{\partial^3 F}{\partial x^3}(0, 0)\left(\frac{\partial F}{\partial x}(0, 0)\right)^3$$

which is negative by assumptions (d) and (f). This completes the proof of the proposition.

There are obvious variants on the theorem which involve changing signs in hypotheses (e) and or (f). Thus we may have an attractive fixed point merging with two repelling points of period two to produce a repelling fixed point, and/or the direction of the bifurcation may be reversed.

2.3 Newton's method and Feigenbaum's constant

Although the values of b_n for the logistic family are hard to compute except by numerical experiment, the superattractive values can be found by applying Newton's method to find the solution, s_n, of the equation

$$L_\mu^{\circ 2^{n-1}}(\frac{1}{2}) = \frac{1}{2}, \quad L_\mu(x) = \mu x(1-x). \tag{2.10}$$

This is the equation for μ which says that $\frac{1}{2}$ is a point of period 2^{n-1} of L_μ. Of course we want to look for solutions for which $\frac{1}{2}$ does not have lower period.

So we set

$$P(\mu) = L_\mu^{2^{n-1}}(\frac{1}{2}) - \frac{1}{2}$$

and apply the Newton algorithm

$$\mu_{k+1} = \mathcal{N}(\mu_k), \qquad \mathcal{N}(\mu) = \mu - \frac{P(\mu)}{P'(\mu)}.$$

with $'$ now denoting differentiation with respect to μ. As a first step, must compute P and P'. For this we define the functions $x_k(\mu)$ recursively by

$$x_0 = \frac{1}{2}, \quad x_1(\mu) = \mu\frac{1}{2}(1-\frac{1}{2}), \quad x_{k+1} = L_\mu(x_k),$$

so, we have

$$\begin{aligned} x'_{k+1} &= [\mu x_k(1-x_k))]' \\ &= x_k(1-x_k) + \mu x'_k(1-x_k) - \mu x_k x'_k \\ &= x_k(1-x_k) + \mu(1-2x_k)x'_k. \end{aligned}$$

Let

$$N = 2^{n-1}$$

so that

$$P(\mu) = x_N - \frac{1}{2}, \quad P'(\mu) = x'_N(\mu).$$

Thus, at each stage of the iteration in Newton's method we compute $P(\mu)$ and $P'(\mu)$ by running the iteration scheme

$$\begin{array}{rclrcl} x_{k+1} &=& \mu x_k(1-x_k) & x_0 &=& \frac{1}{2} \\ x'_{k+1} &=& x_k(1-x_k) + \mu(1-2x_k)x'_x & x'_0 &=& 0 \end{array}$$

for $k = 0, \ldots, N-1$. We substitute this into Newton's method, get the next value of μ, run the iteration to get the next value of $P(\mu)$ and $P'(\mu)$ etc.

Suppose we have found $s_1, s_2, ..., s_n$. What should we take as the initial value of μ? Define the numbers δ_n, $n \geq 2$ recursively by $\delta_2 = 4$ and

$$\delta_n = \frac{s_{n-1} - s_{n-2}}{s_n - s_{n-1}}, \qquad n \geq 3. \tag{2.11}$$

We have already computed

$$s_1 = 2, \quad s_2 = 1 + \sqrt{5} = 3.23606797\ldots.$$

We take as our initial value in Newton's method for finding s_{n+1} the value

$$\mu_{n+1} = s_n + \frac{s_n - s_{n-1}}{\delta_n}.$$

The following facts are observed:

For each $n = 3, 4, \ldots, 15$, Newton's method converges very rapidly, with no changes in the first nineteen digits after six applications of Newton's method for finding s_3, after only one application of Newton's method for s_4 and s_5, and at most four applications of Newton's method for the computation of each of the remaining values.

Suppose we stop our calculations for each s_n when there is no further change in the first 19 digits, and take the computed values as our s_n. These values are strictly increasing. In particular this implies that the s_n we have computed do not yield $\frac{1}{2}$ as a point of lower period.

The s_n approach a limiting value, 3.569945671205296863.

The δ_n approach a limiting value,

$$\delta = 4.6692016148.$$

This value is known as *Feigenbaum's constant.* While the limiting value of the s_n is particular to the logistic family, δ is "universal" in the sense that it applies to a whole class of one dimensional iteration families. We shall go into this point in the next section, where we will see that this is a renormalization group phenomenon.

2.4 Feigenbaum renormalization.

We have already remarked that the rate of convergence to the limiting value of the superstable points in the period doubling bifurcation, Feigenbaum's constant, is universal, i.e. not restricted to the logistic family. That is, if we let

$$\delta = 4.6692....$$

denote Feigenbaum's constant, then the superstable values s_r in the period doubling scenario satisfy

$$s_r = s_\infty - B\delta^{-r} + o(\delta^{-r})$$

where s_∞ and B depend on the specifics of the family, but δ applies to a large class of such families.

There is another "universal" parameter in the story. Suppose that our family f_μ consists of maps with a single maximum, X_m, so that X_m must be one of the points on any superstable periodic orbit. (In the case of the logistic family

$X_m = \frac{1}{2}$.) Let d_r denote the difference between X_m an the next nearest point on the superstable 2^r orbit; more precisely, define

$$d_r = f_{s_r}^{2^{r-1}}(X_m) - X_m.$$

Then

$$d_r \sim D(-\alpha)^r$$

where

$$\alpha \doteq 2.5029...$$

is again universal. This would appear to be a scale parameter (in x) associated with the period doubling scenario. To understand this scale parameter, examine the central portion of Fig 2.4 and observe that the graph of $L_\mu^{\circ 2}$ looks like an (inverted and) rescaled version of L_μ, especially if we allow a change in the parameter μ. The rescaling is centered at the maximum, so in order to avoid notational complexity, let us shift this maximum (for the logistic family) to the origin by replacing x by $y = x - \frac{1}{2}$. In the new coordinates the logistic map is given by

$$y \mapsto L_\mu(y + \frac{1}{2}) - \frac{1}{2} = \mu(\frac{1}{4} - y^2) - \frac{1}{2}.$$

Let $\mathcal{R}$ denote the operator on functions given by

$$\mathcal{R}(h)(y) := -\alpha h(h(y/\alpha)). \tag{2.12}$$

In other words, $\mathcal{R}$ sends a map h into its iterate $h \circ h$ followed by a rescaling. We are going to not only apply the operator $\mathcal{R}$, but also shift the parameter μ in the maps

$$h_\mu(y) = \mu(\frac{1}{2} - y^2) - \frac{1}{2}$$

from one supercritical value to the next. So for each $k = 0, 1, 2, \ldots$ we set

$$g_{k0} := h_{s_k}$$

and then define

$$\begin{aligned} g_{k,1} &= \mathcal{R}g_{k0} \\ g_{k2} &= \mathcal{R}g_{k1} \\ g_{k3} &= \mathcal{R}g_{k2} \\ &\vdots \end{aligned}$$

It is observed (numerically) that for each k the functions g_{kr} appear to be approaching a limit, g_k i.e.

$$g_{kr} \to g_k.$$

So

$$g_k(y) = \lim(-\alpha)^r g_{s_{k+r}}^{2^r}(y/(-\alpha)^r).$$

Hence

$$\mathcal{R}g_k = \lim(-\alpha)^{r+1}2^{r+1}g_{s_{k+r}}(y/(-\alpha)^{r+1}) = g_{k-1}.$$

It is also observed that these limit functions g_k themselves are approaching a limit:

$$g_k \to g.$$

Since $\mathcal{R}g_k = g_{k-1}$ we conclude that

$$\mathcal{R}g = g,$$

i.e. g is a fixed point for the Feigenbaum renormalization operator $\mathcal{R}$. Notice that rescaling commutes with $\mathcal{R}$: If S denotes the operator $(Sf)(y) = cf(y/c)$ then

$$\mathcal{R}(Sf)(y) = -\alpha(c(f(cf(y/(c\alpha))/c) = S(\mathcal{R})f(y).$$

So if g is a fixed point, so is Sg. We may thus fix the scale in g by requiring that

$$g(0) = 1.$$

The hope was then that there would be a unique function g (within an appropriate class of functions) satisfying

$$\mathcal{R}g = g, \quad g(0) = 1,$$

or, spelling this out,

$$g(y) = -\alpha g^{\circ 2}(-y/\alpha), \quad g(0) = 1. \qquad (2.13)$$

Notice that if we knew the function g, then setting $y = 0$ in (2.13) gives

$$1 = -\alpha g(1)$$

or

$$\alpha = -1/g(1).$$

In other words, assuming that we were able to establish all these facts and also knew the function g, then the universal rescaling factor α would be determined by g itself. Feigenbaum assumed that g has a power series expansion in x^2 took the first seven terms in this expansion and substituted in (2.13). He obtained a collection of algebraic equations which he solved and then derived α close to the observed "experimental" value. Indeed, if we truncate (2.13) we will get a collection of algebraic equations. But these equations are not recursive, so that at each stage of truncation modification is made in all the coefficients, and also the nature of the solutions of these equations is not transparent. So theoretically, if we could establish the existence of a unique solution to (2.13) within a given class of functions the value of α is determined. But the numerical evaluation of α is achieved by the renormalization property itself, rather than from $g(1)$ which is not known explicitly.

The other universal constant associated with the period doubling scenario, the constant δ was also conjectured by Feigenbaum to be associated to the fixed point g of the renormalization operator; this time with the linearized map J, i.e. the derivative of the renormalization operator at its fixed point. Later on we will see that in finite dimensions, if the derivative J of a non-linear transformation R at a fixed point has k eigenvalues > 1 in absolute value, and the rest < 1 in absolute value, then there exists a k-dimensional R invariant surface tangent at the fixed point to the subspace corresponding to the k eigenvalues whose absolute value is > 1. On this invariant manifold, the map R is expanding. Feigenbaum conjectured that for the operator $\mathcal{R}$ (acting on the appropriate infinite dimensional space of functions) there is a one dimensional "expanding" submanifold, and that δ is the single eigenvalue of J with absolute value greater than 1.

In the course of the past twenty years, these conjectures of Feigenbaum have been verified using high powered techniques from complex analysis, thanks to the combined effort of such mathematicians as Douady, Hubbard, Sullivan, McMullen, and $\dots$.

2.5 Period 3 implies all periods

Throughout the following f will denote a continuous function on the reals whose domain of definition is assumed to include the given intervals in the various statements.

Lemma 2.5.1 *If $I = [a, b]$ is a compact interval and $I \subset f(I)$ then f has a fixed point in I.*

Proof. For some $c, d \in I$ we have $f(c) = a, f(d) = b$. So $f(c) \leq c, f(d) \geq d$. So $f(x) - x$ changes sign from c to d hence has a zero in between. QED.

Lemma 2.5.2 *If J and $K = [a, b]$ are compact intervals with $K \subset f(J)$ then there is a compact subinterval $L \subset J$ such that $f(L) = K$.*

Proof. Let c be the greatest point in J with $f(c) = a$. If $f(x) = b$ for some $x > c, x \in J$ let d be the least. Then we may take $L = [c, d]$. If not, $f(x) = b$ for some $x < c, x \in J$. Let c' be the largest. Let d' be the the smallest x satisfying $x > c'$ with $f(x) = a$. Notice that $d' \leq c$. We then take $L = [c', d']$. QED

Notation. If I is a closed interval with end points a and b we write

$$I = < a, b >$$

when we do not want to specify which of the two end points is the larger.

Theorem 2.5.1 Sarkovsky *Period three implies all periods.*

Suppose that f has a 3-cycle

$$a \mapsto b \mapsto c \mapsto a \mapsto \cdots .$$

Let a denote the leftmost of the three, and let us assume that

$$a < b < c.$$

(Reversing left and right (i.e. changing direction on the real line) and cycling through the points makes this assumption harmless.) Let

$$I_0 = [a, b], \quad I_1 = [b, c]$$

so we have

$$f(I_0) \supset I_1, \quad f(I_1) \supset I_0 \cup I_1.$$

By Lemma 2 the fact that $f(I_1) \supset I_1$ implies that there is a compact interval $A_1 \subset I_1$ with $f(A_1) = I_1$. Since $f(A_1) = I_1 \supset A_1$ there is a compact subinterval $A_2 \subset A_1$ with $f(A_2) = A_1$. So

$$A_2 \subset A_1 \subset I, \quad f^{\circ 2}(A_2) = I_1.$$

By induction proceed to find compact intervals with

$$A_{n-2} \subset A_{n-3} \subset \cdots \subset A_2 \subset A_1 \subset I_1$$

with

$$f^{\circ(n-2)}(A_{n-2}) = I_1.$$

Since $f(I_0) \supset I_1 \supset A_{n-2}$ there is an interval $A_{n-1} \subset I_0$ with $f(A_{n-1}) = A_{n-2}$. Finally, since $f(I_1) \supset I_0$ there is a compact interval $A_n \subset I_1$ with $f(A_n) = A_{n-1}$. So we have

$$A_n \to A_{n-1} \to \cdots \to A_1 \to I_1$$

where each interval maps onto the next and $A_n \subset I_1$. By Lemma 1, f^n has a fixed point, x, in A_n. But $f(x)$ lies in I_0 and all the higher iterates up to n lie in I_1 so the period can not be smaller than n. So there is a periodic point of any period $n \geq 3$.

Since $f(I_1) \supset I_1$ there is a fixed point in I_1, and since $f(I_0) \supset I_1, f(I_1) \supset I_0$ there is a point of period two in I_0 which is not a fixed point of f. QED

A more refined analysis which we will omit shows that period 5 implies the existence of all periods greater than 5 and period 2 and 4 (but not period 3). In general any odd period implies the existence of periods of all higher order (and all smaller even order). It is easy to graph the third iterate of the logistic map to see that it crosses the diagonal for $\mu > 1 + \sqrt(8)$. In fact, one can prove that that at $\mu = 1 + \sqrt(8)$ the graph of $L_\mu^{\circ 3}$ just touches the diagonal and strictly crosses it for $\mu > 1 + \sqrt(8) = 3.8284\ldots$. Hence in this range there are periodic points of all periods.

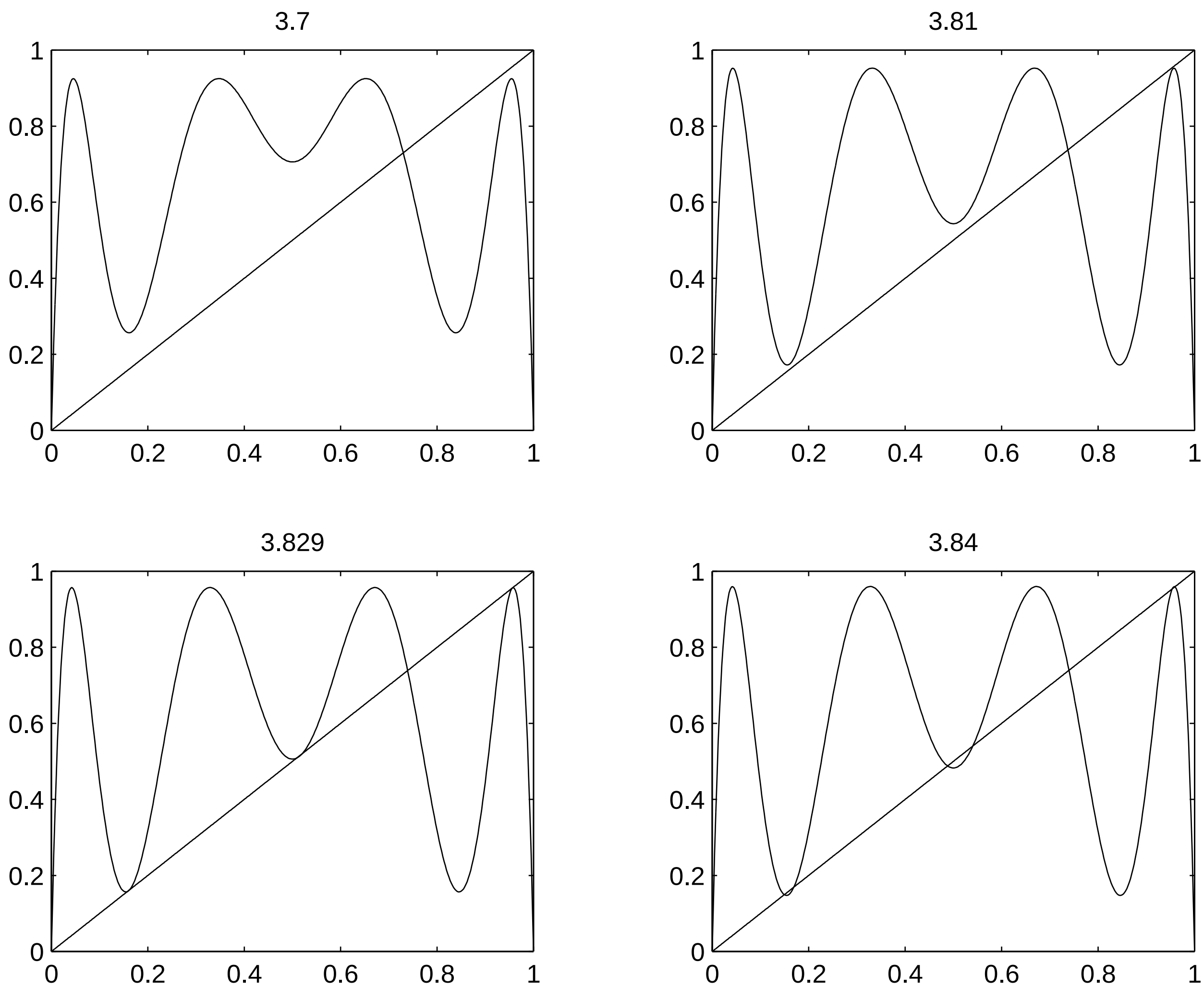

Figure 2.8: Plots of L_μ for $\mu = 3.7, 3.81, 3.83, 3.84$

2.6 Intermittency.

In this section we describe what happens to the period three orbits as we decrease μ from slightly above the critical value $1+\sqrt{8}$ to slightly below it. For $\mu = 1+\sqrt{8}+.002$ the roots of $P(x)-x$ where $P := L_\mu^{\circ 3}$ and the values of P' at these roots are given by

$x =$ roots of $P(x) - x$	$P'(x)$
0	56.20068544683054
0.95756178779471	0.24278522730018
0.95516891475013	1.73457935568109
0.73893250871724	−6.13277919589328
0.52522791460709	1.73457935766594
0.50342728916956	0.24278522531345
0.16402371217410	1.73457935778151
0.15565787278717	0.24278522521922

We see that there is a stable period three orbit consisting of the points

$$0.1556\ldots, .5034\ldots, .9575\ldots.$$

If we choose our initial value of x close to $.5034\ldots$ and plot the successive 199 iterates of L_μ applied to x we obtain the upper graph in Fig. 2.9. The lower graph gives $x(j+3)-x(j)$ for $j=1$ to 197.

We will now decrease the parameter μ by .002 so that $\mu = 1+\sqrt{8}$ is the parameter giving the onset of period three. For this value of the parameter, the graph of $P = L_\mu^{\circ 3}$ just touches the line $y = x$ at the three double roots of $P(x)-x$ which are at $0.1599288\ldots$, $0.514355\ldots$., $0.9563180\ldots$. (Of course, the eighth degree polynomial $P(x)-x$ has two additional roots which correspond to the two (unstable) fixed points of L_μ; these are not of interest to us.) Since the graph of P is tangent to the diagonal at the double roots, $P'(x) = 1$ at these points, so the period three orbit is not strictly speaking stable. But using the same initial seed as above, we do get slow convergence to the period three orbit, as is indicated by Fig. 2.10.

Most interesting is what happens just before the onset of the period three cycle. Then $P(x)-x$ has only two real roots corresponding to the fixed points of L_μ. The remaining six roots are complex. Nevertheless, if μ is close to $1+\sqrt{8}$ the effects of these complex roots can be felt. In Fig. 2.11 we have taken $\mu = 1+\sqrt{8}-.002$ and used the same initial seed $x = .5034$ and again plotted the successive 199 iterates of L_μ applied to x in the upper graph. Notice that there are portions of this graph where the behavior is almost as if we were at a point of period three, followed by some random looking behavior, then almost period three again and so on. This is seen more clearly in the bottom graph of $x(j+3)-x(j)$. Thus the bottom graphs indicates that the deviation from period three is small on the j intervals $j = [1,20]$, $[41,65]$, $[96,108]$, $[119,124]$, $[148,159]$, $[190,?]$.

This phenomenon is known as *intermittency.* We can understand how it works by iterating $P = L_\mu^{\circ 3}$. As we pass close to a minimum of P lying just

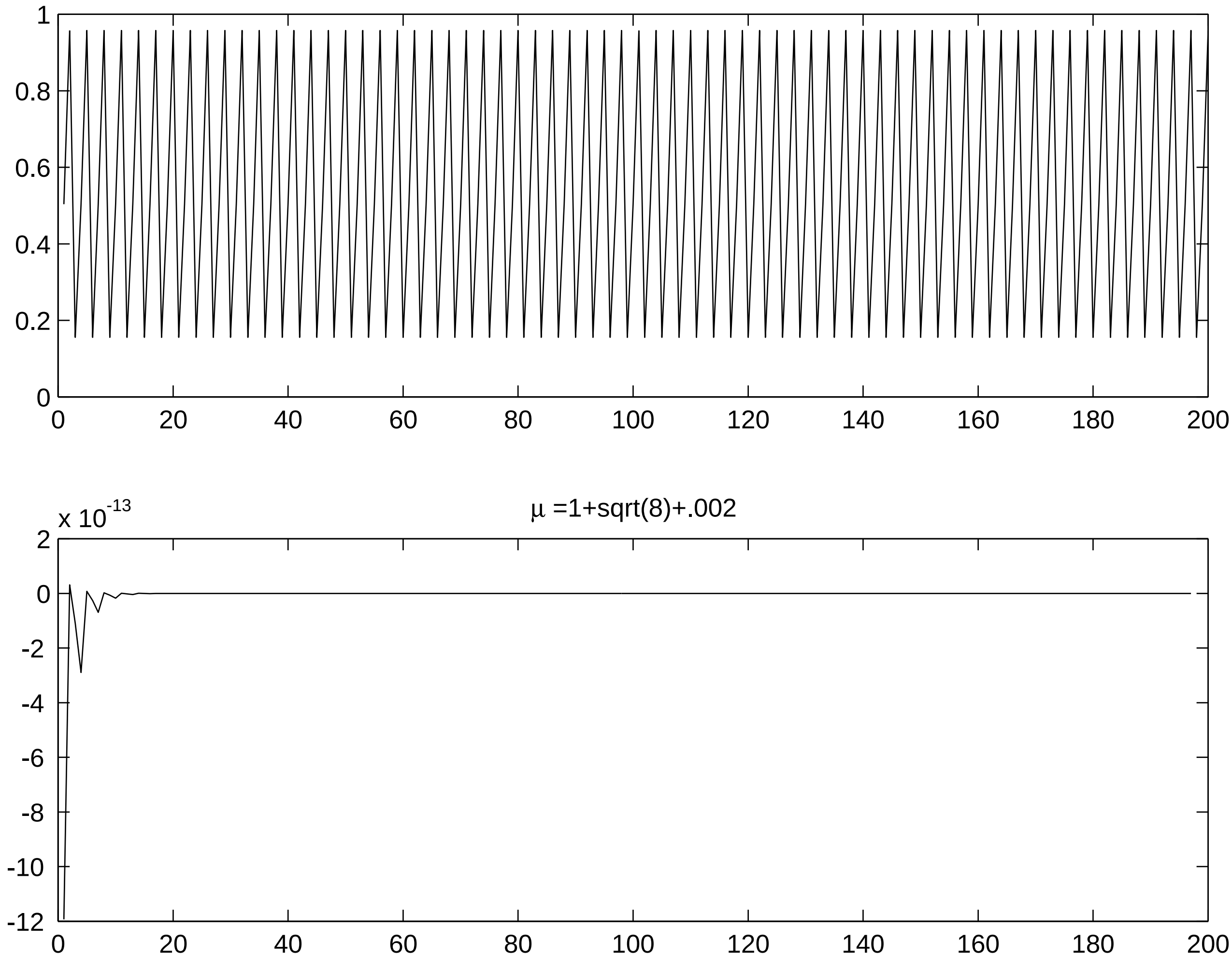

Figure 2.9: $\mu = 1 + \sqrt{8} + .002$

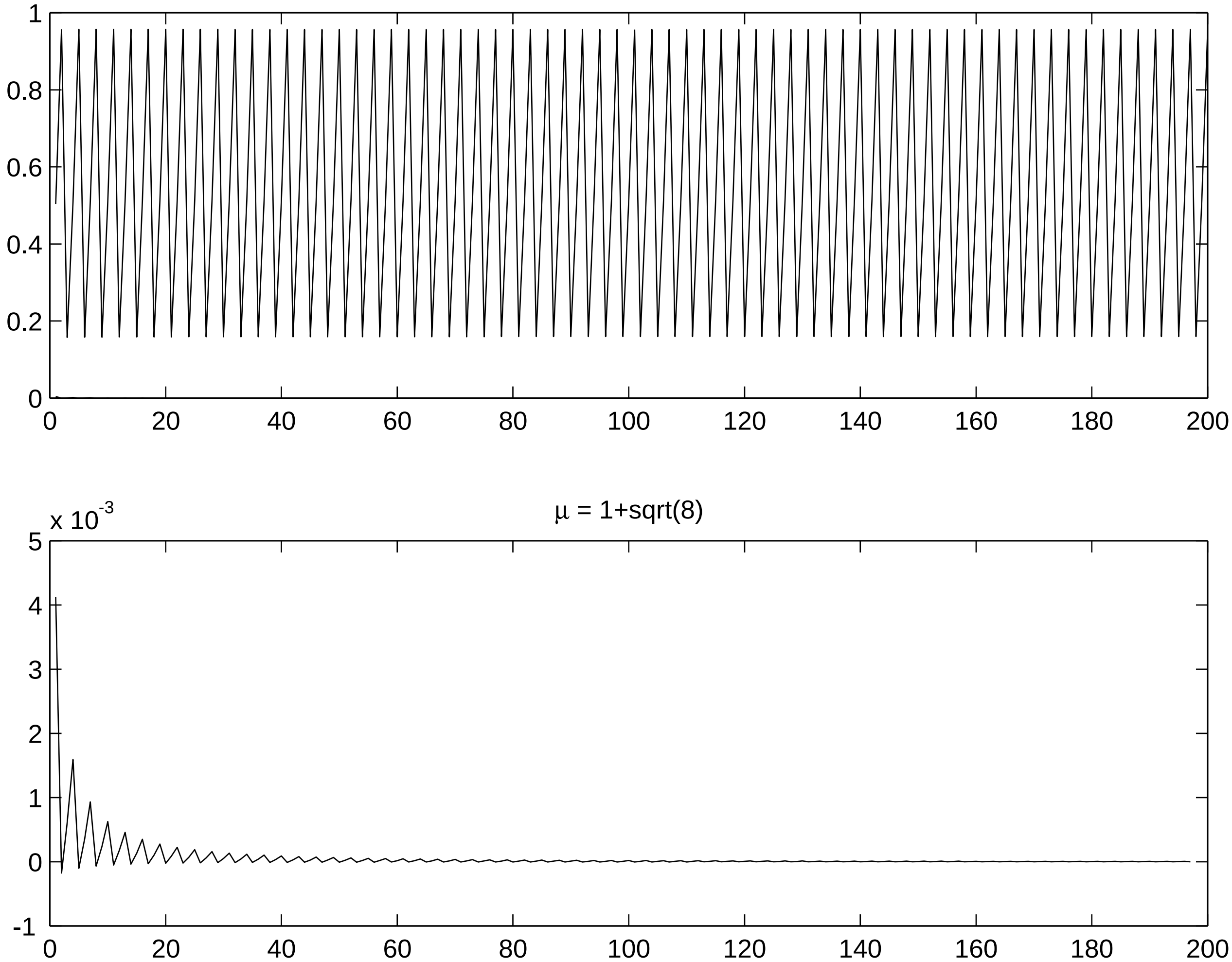

Figure 2.10: $\mu = 1 + \sqrt(8)$

above the diagonal, or to a maximum of P lying just below the diagonal it will take many iterative steps to move away from this region - known as a *bottleneck*. Each such step corresponds to an almost period three cycle. After moving away from these bottlenecks, the steps will be large, eventually hitting a bottleneck once again. See Figures 2.12 and 2.13.

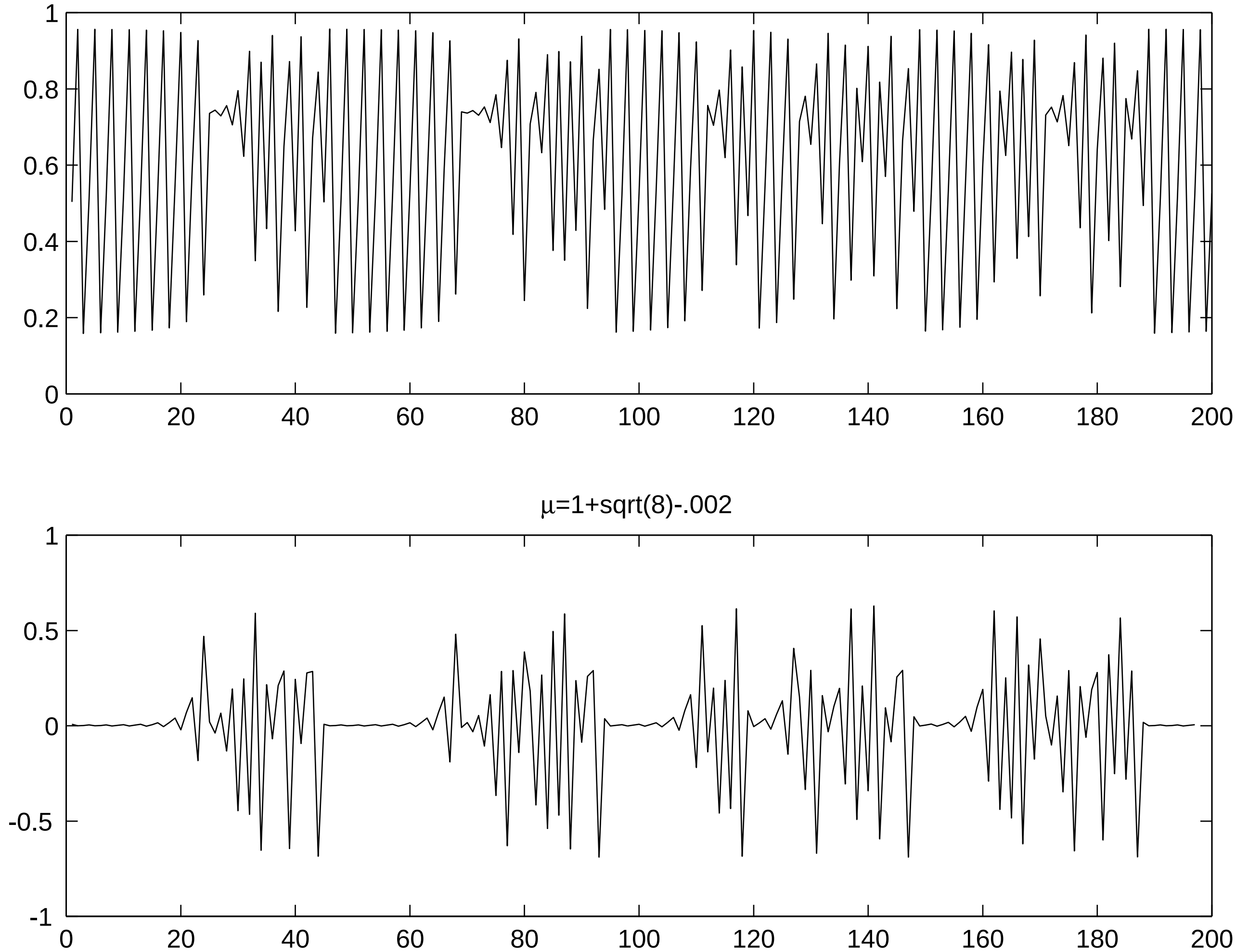

Figure 2.11: $\mu = 1 + \sqrt(8) - .002$

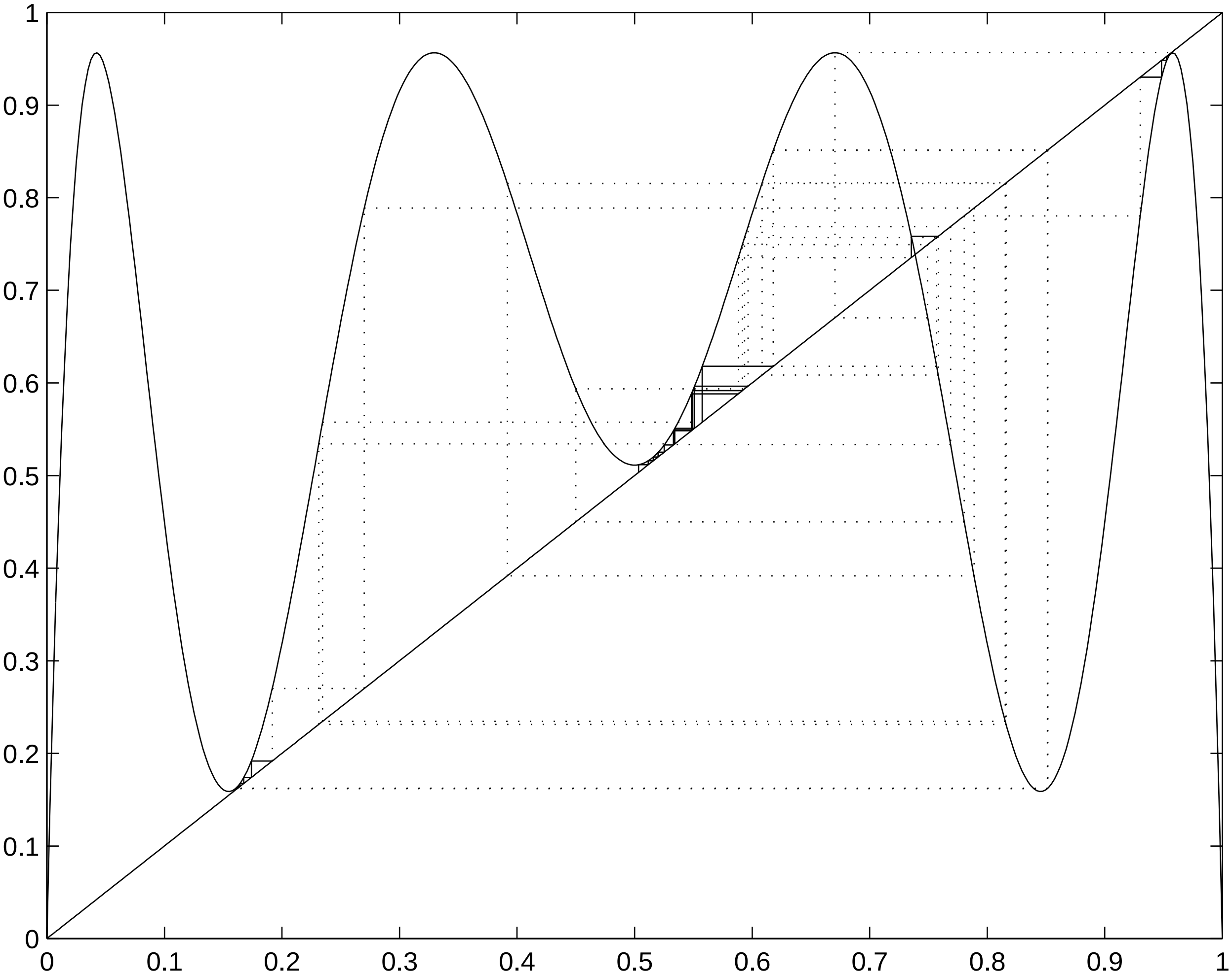

Figure 2.12: Graphical iteration of $P = L_\mu^{\circ 3}$ with $\mu = 1 + \sqrt{8} - .002$ and initial point .5034. The solid lines are iteration steps of size less than .07 representing bottleneck steps. The dotted lines are the longer steps.

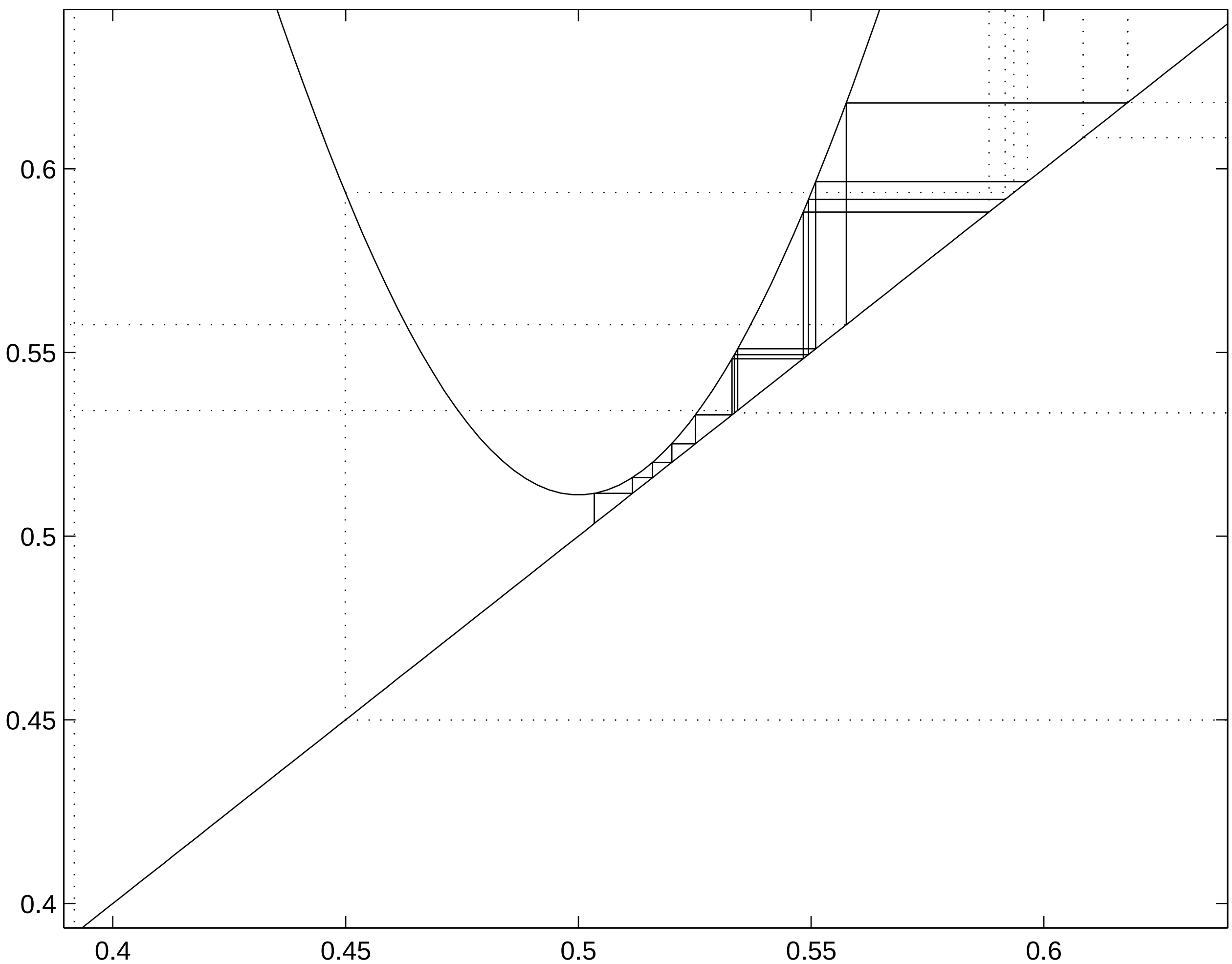

Figure 2.13: Zooming in on the central portion of the preceding figure.