Math 118, Spring 2,001

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March 13, 2001

Contents

Chapter 7

Hyperbolicity.

7.1 C^0 linearization near a hyperbolic point

Let E be a Banach space. A linear map

$$A : E \to E$$

is called *hyperbolic* if we can find closed subspaces S and U of E which are invariant under A such that we have the direct sum decomposition

$$E = S \oplus U \tag{7.1}$$

and a positive constant $a < 1$ so that the estimates

$$\|A_s\| \leq a < 1, \quad A_s = A_{|S} \tag{7.2}$$

and

$$\|A_u^{-1}\| \leq a < 1, \quad A_u = A_{|U} \tag{7.3}$$

hold. (Here, as part of hypothesis (7.3), it is assumed that the restriction of A to U is an isomorphism so that A_u^{-1} is defined.)

If p is a fixed point of a diffeomorphism f, then it is called a *hyperbolic* fixed point if the linear transformation df_p is hyperbolic.

The main purpose of this section is prove that any diffeomorphism, f is conjugate via a local *homeomorphism* to its derivative, df_p near a hyperbolic fixed point. A more detailed statement will be given below. We discussed the one dimensional version of this in Chapter 3.

Proposition 7.1.1 *Let A be a hyperbolic isomorphism (so that A^{-1} is bounded) and let*

$$\epsilon < \frac{1-a}{\|A^{-1}\|}. \tag{7.4}$$

If ϕ and ψ are bounded Lipschitz maps of E into itself with

$$\mathrm{Lip}[\phi] < \epsilon, \quad \mathrm{Lip}[\psi] < \epsilon$$

then there is a unique solution to the equation

$$(\mathrm{id} + u) \circ (A + \phi) = (A + \psi) \circ (\mathrm{id} + u) \tag{7.5}$$

in the space, X of bounded continuous maps of E into itself. If $\phi(0) = \psi(0) = 0$ then $u(0) = 0$.

Proof. If we expand out both sides of (7.5) we get the equation

$$Au - u(A + \phi) = \phi - \psi(\mathrm{id} + u).$$

Let us define the linear operator, L, on the space X by

$$L(u) = Au - u \circ (\mathrm{id} + \phi).$$

So we wish to solve the equation

$$L(u) = \phi - \psi(A + u).$$

We shall show that L is invertible with

$$\|L^{-1}\| \leq \frac{\|A^{-1}\|}{(1-a)}. \tag{7.6}$$

Assume, for the moment that we have proved (7.6). We are then looking for a solution of

$$u = K(u)$$

where

$$K(u) = L^{-1}[\phi - \psi(\mathrm{id} + u)].$$

But

$$\begin{aligned} \|K(u_1) - K(u_2)\| &= \|L^{-1}[\phi - \psi(\mathrm{id} + u_1) - \phi + \psi(\mathrm{id} + u_2)]\| \\ &= \|L^{-1}[\psi(\mathrm{id} + u_2) - \psi(\mathrm{id} + u_1)]\| \\ &\leq \|L^{-1}\| \mathrm{Lip}[\psi] \|u_2 - u_1\| \\ &< c\|u_2 - u_1\|, \quad c < 1 \end{aligned}$$

if we combine (7.6) with (7.4). Thus K is a contraction and we may apply the contraction fixed point theorem to conclude the existence and uniqueness of the solution to (7.5). So we turn our attention to the proof that L is invertible and of the estimate (7.6). Let us write

$$Lu = A(Mu)$$

where

$$Mu = u - A^{-1} u \circ (A + \phi).$$

Composition with A is an invertible operator and the norm of its inverse is $\|A^{-1}\|$. So we are reduced to proving that M is invertible and that we have the estimate

$$\|M^{-1}\| \leq \frac{1}{1-a}. \tag{7.7}$$

Let us write

$$u = f \oplus g, \quad f : E \to S, \quad g : E \to U$$

in accordance with the decomposition (7.1). So if we let Y denote the space of bounded continuous maps from E to S, and let Z denote the space of bounded continuous maps from E to U, we have

$$X = Y \oplus Z$$

and the operator M sends each of the spaces Y and Z into themselves since A^{-1} preserves S and U. We let M_s denote the restriction of M to Y, and let M_u denote the restriction of M to Z. It will be enough for us to prove that each of the operators M_s and M_u is invertible with a bounds (7.7) with M replaced by M_s and by M_u. For $f \in Y$ let us write

$$M_s f = f - Nf, \quad Nf = A^{-1} f \circ (A + \phi).$$

We will prove

Lemma 7.1.1 *The map N is invertible and we have*

$$\|N^{-1}\| \leq a.$$

Proof. We claim that he map $A + \phi$ a homeomorphism with Lipschitz inverse. Indeed

$$\|Ax\| \geq \frac{1}{\|A^{-1}\|}\|x\|$$

so

$$\begin{aligned} \|Ax + \phi(x) - Ay - \phi(y)\| &\geq \left[\frac{1}{\|A^{-1}\|} - \mathrm{Lip}[\phi]\right]\|x - y\| \\ &\geq \frac{a}{\|A^{-1}\|}\|x - y\| \end{aligned}$$

by (7.4). This shows that $A + \phi$ is one to one. Furthermore, to solve

$$Ax + \phi(x) = y$$

for x, we apply the contraction fixed point theorem to the map

$$x \mapsto A^{-1}(y - \phi(x)).$$

The estimate (7.4) shows that this map is a contraction. Hence $A + \phi$ is also surjective.

Thus the map N is invertible, with

$$N^{-1}f = A_s f \circ (A+\phi)^{-1}.$$

Since $\|A_s\| \le a$, we have

$$\|N^{-1}f\| \le a\|f\|.$$

(This is in terms of the sup norm on Y.) In other words, in terms of operator norms,

$$\|N^{-1}\| \le a.$$

We can now find M_s^{-1} by the geometric series

$$\begin{aligned} M_s^{-1} &= (I-N)^{-1} \\ &= [(-N)(I-N^{-1})]^{-1} \\ &= (-N)^{-1}[I+N^{-1}+N^{-2}+N^{-3}+\cdots] \end{aligned}$$

and so on Y we have the estimate

$$\|M_s^{-1}\| \le \frac{a}{1-a}.$$

The restriction, M_u, of M to Z is

$$M_u g = g - Qg$$

with

$$\|Qg\| \le a\|g\|$$

so we have the simpler series

$$M_u^{-1} = I + Q + Q^2 + \cdots$$

giving the estimate

$$\|M_u\| \le \frac{1}{1-a}.$$

Since

$$\frac{a}{1-a} < \frac{1}{1-a}$$

the two pieces together give the desired estimate

$$\|M\| \le \frac{1}{1-a},$$

completing the proof of the first part of the proposition. Since evaluation at zero is a continuous function on X, to prove the last statement of the proposition it is enough to observe that if we start with an initial approximation satisfying $u(0) = 0$ (for example $u \equiv 0$) Ku will also satisfy this condition and hence so will $K^n u$ and therefor so will the unique fixed point.

Now let f be a differentiable, hyperbolic transformation defined in some neighborhood of 0 with $f(0) = 0$ and $df_0 = A$. We may write

$$f = A + \phi$$

where

$$\phi(0) = 0, \quad d\phi_0 = 0.$$

We wish to prove

Theorem 7.1.1 *There exists neighborhoods U and V of 0 and a homeomorphism $h : U \to V$ such that*

$$h \circ A = f \circ h. \tag{7.8}$$

We prove this theorem by modifying ϕ outside a sufficiently small neighborhood of 0 in such a way that the new ϕ is globally defined and has Lipschitz constant less than ϵ where ϵ satisfies condition (7.4). We can then apply the proposition to find a global h which conjugates the modified f to A, and $h(0) = 0$. But since we will not have modified f near the origin, this will prove the local assertion of the theorem. For this purpose, choose some function $\rho : \mathbf{R} \to \mathbf{R}$ with

$$\begin{aligned} \rho(t) &= 0 \qquad \forall \quad t \geq 1 \\ \rho(t) &= 1 \qquad \forall \quad t \leq \frac{1}{2} \\ |\rho'(t)| &< K \quad \forall t \end{aligned}$$

where K is some number,

$$K > 2.$$

For a fixed ϵ let r be sufficiently small so that the on the ball, $B_r(0)$ we have the estimate

$$\|d\phi_x\| < \frac{\epsilon}{2K},$$

which is possible since $d\phi_0 = 0$ and $d\phi$ is continuous. Now define

$$\psi(x) = \rho(\frac{\|x\|}{r})\phi(x),$$

and continuously extend to

$$\psi(x) = 0, \quad \|x\| \geq r.$$

Notice that

$$\psi(x) = \phi(x), \quad \|x\| \leq \frac{r}{2}.$$

Let us now check the Lipschitz constant of ψ. There are three alternatives: If x_1 and x_2 both belong to $B_r(0)$ we have

$$\begin{aligned} \|\psi(x_1) - \psi(x_2)\| &= \|\rho(\frac{\|x_1\|}{r})\phi(x_1) - \rho(\frac{\|x_2\|}{r})\phi(x_2) \\ &\leq |\rho(\frac{\|x_1\|}{r}) - \rho(\frac{\|x_2\|}{r})|\|\phi(x_1)\| + \rho(\frac{\|x_2\|}{r})\|\phi(x_1) - \phi(x_2)\| \\ &\leq (K\|x_1 - x_2\|/r) \times \|x_1\| \times (\epsilon/2K) + (\epsilon/2K) \times \|x_1 - x_2\| \\ &\leq \epsilon\|x_1 - x_2|. \end{aligned}$$

If $x_1 \in B_r(0),\ x_2 \notin B_r(0)$, then the second term in the expression on the second line above vanishes and the first term is at most $(\epsilon/2)\|x_1 - x_2\|$. If neither x_1 nor x_2 belong to $B_r(0)$ then $\psi(x_1) - \psi(x_2) = 0 - 0 = 0$. We have verified that $\mathrm{Lip}[\psi] < \epsilon$ and so have proved the theorem.

7.2 invariant manifolds

Let p be a hyperbolic fixed point of a diffeomorphism, f. The *stable manifold* of f at p is defined as the set

$$W^s(p) = W^s(p, f) = \{x | \lim_{n\to\infty} f^n(x) = p\}. \tag{7.9}$$

Similarly, the *unstable manifold* of f at p is defined as

$$W^u(p) = W^u(p, f) = \{x | \lim_{n\to\infty} f^{-n}(x) = p\}. \tag{7.10}$$

We have defined W^s and W^u as sets. We shall see later on in this section that in fact they are submanifolds, of the same degree of smoothness as f. The terminology, while standard, is unfortunate. A point which is not exactly on $W^s(p)$ is swept away under iterates of f from any small neighborhood of p. This is the content of our first proposition below. So it is a very *unstable* property to lie on W^s. Better terminology would be "contracting" and "expanding" submanifolds. But the usage is standard, and we will abide by it. In any event, the sets $W^s(p)$ and $W^u(p)$ are, by their very definition, invariant under f.

In the case that $f = A$ is a hyperbolic *linear* transformation on a Banach space $E = S \oplus U$, then $W^s(0) = S$ and $W^u(0) = U$ as follows immediately from the definitions. The main result of this section will be to prove that in the general case, the stable manifold of f at p will be a submanifold whose tangent at p is the stable subspace of the linear transformation df_p.

Notice that for a hyperbolic fixed point, replacing f by f^{-1} interchanges the roles of W^s and W^u. So in much of what follows we will formulate and prove theorems for either W^s or for W^u. The corresponding results for W^u or for W^s then follow automatically.

Let A be a hyperbolic linear transformation on a Banach space $E = S \oplus U$, and consider any ball, $B_r = B_r(0)$ of radius r about the origin. If $x \in B_r$ does *not* lie on $S \cap B_r$, this means that if we write $x = x_s \oplus x_u$ with $x_s \in S$ and $x_u \in U$ then $x_u \neq 0$. Then

$$\begin{aligned}
\|A^n x\| &= \|A^n x_s\| + \|A^n x_u\| \\
&\geq \|A^n x_u\| \\
&\geq c^n \|x_u\|.
\end{aligned}$$

If we choose n large enough, we will have $c^n\|x_u\| > r$. So eventually, $A^n x \notin B_r$. Put contrapositively,

$$S \cap B_r = \{x \in B_r | A^n x \in B_r \forall n \geq 0\}.$$

Now consider the case of a hyperbolic fixed point, p, of a diffeomorphism, f. We may introduce coordinates so that $p = 0$, and let us take $A = df_0$. By the C^0 conjugacy theorem, we can find a neighborhood, V of 0 and homeomorphism

$$h : B_r \to V$$

with

$$h \circ f = A \circ h.$$

Then

$$f^n(x) = h^{-1} \circ A^n \circ h\ (x)$$

will lie in U for all $n \geq 0$ if and only if $h(x) \in S(A)$ if and only if $A^n h(x) \to 0$. This last condition implies that $f^n(x) \to p$. We have thus proved

Proposition 7.2.1 *Let p be a hyperbolic fixed point of a diffeomorphism, f. For any ball, $B_r(p)$ of radius r about p, let*

$$B_r^s(p) = \{x \in B_r(p) | f^n(x) \in B_r^s(p) \forall n \geq 0\}. \tag{7.11}$$

Then for sufficiently small r, we have

$$B_r^s(p) \subset W^s(p).$$

Furthermore, our proof shows that for sufficiently small r the set $B_r^s(p)$ is a topological submanifold in the sense that every point of $B_r^s(p)$ has a neighborhood (in $B_r^s(p)$) which is the image of a neighborhood, V in a Banach space under a homeomorphism, H. Indeed, the restriction of h to S gives the desired homeomorphism.

Remark. In the general case we can not say that $B_r^s(p) = B_r(p) \cap W^s(p)$ because a point may escape from $B_r(p)$, wander around for a while, and then be drawn towards p.

But the proposition does assert that $B_r^s(p) \subset W^s(p)$ and hence, since W^s is invariant under f^{-1}, we have

$$f^{-n}[B_r^s(p)] \subset W^s(p)$$

for all n, and hence

$$\bigcup_{n \geq 0} f^{-n}[B_r^s(p)] \subset W^s(p).$$

On the other hand, if $x \in W^s(p)$, which means that $f^n(x) \to p$, eventually $f^n(x)$ arrives and stays in any neighborhood of p. Hence $p \in f^{-n}[B_r^s(p)]$ for some n. We have thus proved that for sufficiently small r we have

$$W^s(p) = \bigcup_{n \geq 0} f^{-n}[B_r^s(p)]. \tag{7.12}$$

We will prove that $B_r^s(p)$ is a submanifold. It will then follow from (7.12) that $W^s(p)$ is a submanifold. The global disposition of $W^s(p)$, and in particular its relation to the stable and unstable manifolds of other fixed points, is a key ingredient in the study of the long term behavior of dynamical systems. In this section our focus is purely local, to prove the smooth character of the set $B_r^s(p)$. We follow the treatment in [**?**].

We will begin with the hypothesis that f is merely Lipschitz, and give a proof (independent of the C^0 linearization theorem) of the existence and Lipschitz character of the W^u. We will work in the following situation: A is a hyperbolic linear isomorphism of a Banach space $E = S \oplus U$ with

$$\|Ax\| \leq a\|x\|, \; x \in S, \quad \|A^{-1}x\| \leq a\|x\|, x \in U.$$

We let $S(r)$ denote the ball of radius s about the origin in S, and $U(r)$ the ball of radius r in U. We will assume that

$$f : S(r) \times U(r) \to E$$

is a Lipschitz map with

$$\|f(0)\| \leq \delta \tag{7.13}$$

and

$$\mathrm{Lip}[f - A] \leq \epsilon. \tag{7.14}$$

We wish to prove the following

Theorem 7.2.1 *Let $c < 1$. There exists an $\epsilon = \epsilon(a)$ and a $\delta = \delta(a, \epsilon, r)$ so that if f satisfies (7.13) and (7.14) then there is a map*

$$g : E_u(r) \to E_s(r)$$

with the following properties:
(i) g is Lipschitz with $\mathrm{Lip}[g] \leq 1$.
(ii) The restriction of f^{-1} to graph(g) is contracting and hence has a fixed point, p, on graph(g).
(iii) We have

$$graph(g) = \bigcap f^n(S(r) \oplus U(r)) = W^u(p) \cap [S(r) \oplus U(p)].$$

The idea of the proof is to apply the contraction fixed point theorem to the space of maps of $U(r)$ to $S(r)$. We want to identify such a map, v, with its graph:

$$graph(v) = \{(v(x), x), \; x \in U(r)\}.$$

Now

$$f[graph(v)] = \{f(v(x), x)\} = \{(f_s(v(x), x), f_u(v(x), x))\},$$

where we have introduced the notation

$$f_s = p_s \circ f, \quad f_u = p_u \circ f,$$

where p_s denotes projection onto S and p_u denotes projection onto U.

Suppose that the projection of $f[graph(v)]$ onto U is injective and its image contains $U(r)$. This means that for any $y \in U(r)$ there is a unique $x \in U(r)$ with

$$f_u(v(x), x) = y.$$

So we write

$$x = [f_u \circ (v, id)]^{-1}(y)$$

where we think of (v, id) as a map of $U(r) \to E$ and hence of

$$f_u \circ (v, id)$$

as a map of $U(r) \to U$. Then we can write

$$f[graph(v)] = \{(f_s(v([f_u \circ (v, id)]^{-1}(y), y)\} = graphG_[f(v)]$$

where

$$G_f(v) = f_s \circ (v, id) \circ [f_u \circ (v, id)]^{-1}. \tag{7.15}$$

The map $v \mapsto G_f(v)$ is called the *graph transform* (when it is defined). We are going to take

$$X = \mathrm{Lip}_1(U(r), S(r))$$

to consist of all Lipschitz maps from $U(r)$ to $S(r)$ with Lipschitz constant ≤ 1. The purpose of the next few lemmas is to show that if ϵ and δ are sufficiently small then the graph transform, G_f is defined and is a contraction on X. The contraction fixed point theorem will then imply that there is a unique $g \in X$ which is fixed under G_f, and hence that $graph(g)$ is invariant under f. We will then find that g has all the properties stated in the theorem.

In dealing with the graph transform it is convenient to use the box metric, $|\ |$, on $S \oplus U$ where

$$|x_s \oplus x_u| = \max\{\|x_s\|, \|x_u\|\}$$

i.e.

$$|x| = \max\{\|p_s(x)\|, \|p_u(x)\|\}.$$

We begin with

Lemma 7.2.1 *If* $v \in X$ *then*

$$\mathrm{Lip}[f_u \circ (v, id) - A_u] \leq \mathrm{Lip}[f - A].$$

Proof. Notice that

$$p_u \circ A(v(x), x) = p_u(A_s(v(x)), A_u x) = A_u x$$

so

$$f_u \circ (v, id) - A_u = p_u \circ [f - A] \circ (v, id).$$

We have $\mathrm{Lip}[p_u] \leq 1$ since p_u is a projection, and

$$\mathrm{Lip}(v, id) \leq \max\{\mathrm{Lip}[v], \mathrm{Lip}[id]\} = 1$$

since we are using the box metric. Thus the lemma follows.

Lemma 7.2.2 *Suppose that* $0 < \epsilon < c^{-1}$ *and*

$$\mathrm{Lip}[f - A] < \epsilon.$$

Then for any $v \in X$ *the map* $f_u \circ (v, id) : E_u(r) \to E_u$ *is a homeomorphism whose inverse is a Lipschitz map with*

$$\mathrm{Lip}\left[[f_u \circ (v, id)]^{-1}\right] \leq \frac{1}{c^{-1} - \epsilon}. \tag{7.16}$$

Proof. Using the preceding lemma, we have

$$\mathrm{Lip}[f_u - A_u] < \epsilon < c^{-1} < \|A_u^{-1}\|^{-1} = (\mathrm{Lip}[A_u])^{-1}.$$

By the Lipschitz implicit function theorem we conclude that $f_u \circ (v, id)$ is a homeomorphism with

$$\mathrm{Lip}\left[[f_u \circ (v, id)]^{-1}\right] \leq \frac{1}{\|A_u^{-1}\|^{-1} - \mathrm{Lip}[f_u \circ (v, id) - A_u]} \leq \frac{1}{c^{-1} - \epsilon}$$

by another application of the preceding lemma. QED. We now wish to show that the image of $f_u \circ (v, id)$ contains $U(r)$ if ϵ and δ are sufficiently small: By the proposition in section 5.2 concerning the image of a Lipschitz map, we know that the image of $U(r)$ under $f_u \circ (v, id)$ contains a ball of radius r/λ about $[f_u \circ (v, id)](0)$ where λ is the Lipschitz constant of $[f_u \circ (v, id)]^{-1}$. By the preceding lemma, $r/\lambda = r(c^{-1} - \epsilon)$. Hence $f_u \circ (v, id)(U(r))$ contains the ball of radius

$$r(c^{-1} - \epsilon) - \|f_u(v(0), 0)\|$$

about the origin. But

$$\begin{aligned}
\|f_u(v(0), 0)\| &\leq \|f_u(0, 0)\| + \|f_u(v(0), 0) - f_u(0, 0)\| \\
&\leq \|f_u(0, 0)\| + \|(f_u - p_u A)(v(0), 0) - (f_u - p_u A)(0, 0)\| \\
&\leq |f(0)| + |(f - A)(v(0), 0) - (f - A)(0, 0)| \\
&\leq |f(0)| + \epsilon r.
\end{aligned}$$

The passage from the second line to the third is because $p_u A(x, y) = A_u y = 0$ if $y = 0$. The passage from the third line to the fourth is because we are using the box norm. So

$$r(c^{-1} - \epsilon) - \|f_u(v(0), 0)\| \geq r(c^{-1} - 2\epsilon) - \delta$$

if (7.13) holds. We would like this expression to be $\geq r$, which will happen if

$$\delta \leq r(c^{-1} - 1 - 2\epsilon). \tag{7.17}$$

We have thus proved

Proposition 7.2.2 *Let f be a Lipschitz map satisfying (7.13) and (7.14) where $2\epsilon < c^{-1}-1$ and (7.17) holds. Then for every $v \in X$, the graph transform, $G_f(v)$ is defined and*

$$\mathrm{Lip}[G_f(v)] \leq \frac{c+\epsilon}{c^{-1}-\epsilon}.$$

The estimate on the Lipschitz constant comes from

$$\begin{aligned}
\mathrm{Lip}[G_f(v)] &\leq \mathrm{Lip}[f_s \circ (v, \mathrm{id})]\mathrm{Lip}[(f_u \circ (v, \mathrm{id})] \\
&\leq \mathrm{Lip}[f_s]\mathrm{Lip}[v]\mathrm{Lip} \cdot \frac{1}{c^{-1}-\epsilon} \\
&\leq (\mathrm{Lip}[A_s] + \mathrm{Lip}[p_s \circ (f-A)]) \cdot \frac{1}{c^{-1}-\epsilon} \\
&\leq \frac{c+\epsilon}{c^{-1}-\epsilon}.
\end{aligned}$$

In going from the first line to the second we have used the preceding lemma.

In particular, if

$$2\epsilon < c^{-1} - c \tag{7.18}$$

then

$$\mathrm{Lip}[G_f(v)] \leq 1.$$

Let us now obtain a condition on δ which will guarantee that

$$G_f(v)(U(r) \subset S(r).$$

Since

$$f_u \circ (v, \mathrm{id})U(r) \supset U(r),$$

we have

$$[f_u \circ (v, \mathrm{id})]^{-1}U(r) \subset U(r).$$

Hence, from the definition of $G_f(v)$, it is enough to arrange that

$$f_s \circ (v, \mathrm{id})[U(r)] \subset S(r).$$

For $x \in U(r)$ we have

$$\begin{aligned}
\|f_s(v(x), x)\| &\leq \|p_s \circ (f-A)(v(x), x)\| + \|A_s v(x)\| \\
&\leq |(f-A)(v(x), x)| + c\|v(x)\| \\
&\leq |(f-A)(v(x), x) - (f-A)(0,0)| + |f(0)| + cr \\
&\leq \epsilon|(v(x), x)| + \delta + cr \\
&\leq \epsilon r + \delta + cr.
\end{aligned}$$

So we would like to have

$$(\epsilon + c)r + \delta < r$$

or

$$\delta \leq r(1-c-\epsilon). \tag{7.19}$$

If this holds, then G_f maps X into X.

We now want conditions that guarantee that G_f is a contraction on X, where we take the sup norm. Let (w,x) be a point in $S(r) \oplus U(r)$ such that $f_u(w,x) \in U(r)$. Let $v \in X$, and consider

$$|(w,x)-(v(x),x)| = \|w-v(x)\|,$$

which we think of as the distance along S from the point (w,x) to $graph(v)$. Suppose we apply f. So we replace (w,x) by $f(w,x) = (f_s(w,x), f_u(w,x))$ and $graph(v)$ by $f(graph(v)) = graph(G_f(v))$. The corresponding distance along S is $\|f_s(w,x) - G_f(v)(f_u(w,x)\|$. We claim that

$$\|f_s(w,x) - G_f(v)(f_u(w,x))\| \leq (c+2\epsilon)\|w-v(x)\|. \tag{7.20}$$

Indeed,

$$f_s(v(x),x) = G_f(v)(f_u(v(x),x)$$

by the definition of G_f, so we have

$$\begin{aligned}
\|f_s(w,x) - G_f(v)(f_u(w,x))\| &\leq \|f_s(w,x) - f_s(v(x),x)\| + \\
&\quad +\|G_f(v)(f_u((v(x),x) - G_f(v)(f_u(w,x))\| \\
&\leq \text{Lip}[f_s]|(w,x)-(v(x),x)| + \\
&\quad +\text{Lip}[f_u]|(v(x),x)-(w,x)| \\
&\leq \text{Lip}[f_s - p_sA + p_sA]\|w-v(x)\| + \\
&\quad +\text{Lip}[f_u - p_uA]\|w-v(x)\| \\
&\leq (\epsilon + c + \epsilon)\|w-v(x)\|
\end{aligned}$$

which is what was to be proved.

Consider two elements, v_1 and v_2 of X. Let z be any point of $U(r)$, and apply (7.20) to the point

$$(w,x) = (v_1([f_u \circ (v_1, \text{id})]^{-1}](z)), [f_u \circ (v_1, \text{id})]^{-1}](z))$$

which lies on $graph(v_1)$, and where we take $v = v_2$ in (7.20). The image of (w,x) is the point $(G_f(v_1)(z), z)$ which lies on $graph(G_f(v_1))$, and, in particular, $f_u(w,x) = z$. So (7.20) gives

$$\|G_f(v_1)(z) - G_f(v_2)(z)\| \leq (c+2\epsilon)\|v_1([f_u \circ (v_1, \text{id})]^{-1}](z)) - v_2([f_u \circ (v_1, \text{id})]^{-1}](z)\|.$$

Taking the sup over z gives

$$\|G_f(v_1) - G_f(v_2)\|_{\text{sup}} \leq (c+2\epsilon)\|v_1 - v_2\|_{\text{sup}}. \tag{7.21}$$

Intuitively, what (7.20) is saying is that G_f multiplies the S distance between two graphs by a factor of at most $(c+2\epsilon)$. So G_f will be a contraction in the sup norm if

$$2\epsilon < 1 - c \tag{7.22}$$

which implies (7.18). To summarize: we have proved that G_f is a contraction in the sup norm on X if (7.17), (7.19) and (7.22) hold, i.e.

$$2\epsilon < 1 - c, \quad \delta < r \min(c^{-1} - 1 - 2\epsilon,\ 1 - c - \epsilon).$$

Notice that since $c < 1$, we have $c^{-1} - 1 > 1 - c$ so both expressions occurring in the min for the estimate on δ are positive.

Now the uniform limit of continuous functions which all have $\mathrm{Lip}[v] \leq 1$ has Lipschitz constant ≤ 1. In other words, X is closed in the sup norm as a subset of the space of continuous maps of $U(r)$ into $S(r)$, and so we can apply the contraction fixed point theorem to conclude that there is a unique fixed point, $g \in X$ of G_f. Since $g \in X$, condition (i) of the theorem is satisfied. As for (ii), let $(g(x), x)$ be a point on $graph(g)$ which is the image of the point $(g(y), y)$ under f, so

$$(g(x), x) = f(g(y), y)$$

which implies that

$$x = [f_u \circ (g, \mathrm{id})](y).$$

We can write this equation as

$$p_u \circ f_{|graph(g)} = [f_u \circ (g, \mathrm{id})] \circ (p_u)_{|graph(g)}.$$

In other words, the projection p_u conjugates the restriction of f to $graph(g)$ into $[f_u \circ (g, \mathrm{id})]$. Hence the restriction off^{-1} to $graph(g)$ is conjugated by p_u into $[f_u \circ (g, \mathrm{id})]^{-1}$. But, by (7.16), the map $[f_u \circ (g, \mathrm{id})]^{-1}$ is a contraction since

$$c^{-1} - 1 > 1 - c > 2\epsilon$$

so

$$c^{-1} - \epsilon > 1 + \epsilon > 1.$$

The fact that $\mathrm{Lip}[g] \leq 1$ implies that

$$|(g(x), x) - (g(y), y)| = \|x - y\|$$

since we are using the box norm. So the restriction of p_u to $graph(g)$ is an isometry between the (restriction of) the box norm on $graph(g)$and the norm on U. So we have proved statement (ii), that the restriction of f^{-1} to $graph(g)$ is a contraction.

We now turn to statement (iii) of the theorem. Suppose that (w, x) is a point in $S(r) \oplus U(r)$ with $f(w, x) \in S(r) \oplus U(r)$. By (7.20) we have

$$\|f_s(w, x) - g(f_u(w, x)\| \leq (c + 2\epsilon)\|w - g(x)\|$$

since $G_f(g) = g$. So if the first n iterates of f applied to (w, x) all lie in $S(r) \oplus U(r)$, and if we write

$$f^n(w, x) = (z, y),$$

we have

$$\|z - g(y)\| \leq (c + 2\epsilon)^n \|w - g(x)\| \leq (c + 2\epsilon)r.$$

So if the point (z, y) is in $\bigcap f^n(S(r) \oplus U(r))$ we must have $z = g(y)$, in other words

$$\bigcap f^n(S(r) \oplus U(r)) \subset graph(g).$$

But

$$graph(g) = f[graph(g)] \cap [S(r) \oplus U(r)]$$

so

$$graph(g) \subset \bigcap f^n(S(r) \oplus U(r)),$$

proving that

$$graph(g) = \bigcap f^n(S(r) \oplus U(r)).$$

We have already seen that the restriction of f^{-1} to $graph(g)$ is a contraction, so all points on $graph(g)$ converge under the iteration of f^{-1} to the fixed point, p. So they belong to $W^u(p)$. This completes the proof of the theorem.

Notice that if $f(0) = 0$, then $p = 0$ is the unique fixed point.