EXISTENCE OF SOLUTIONS TO ODE's — Math118, O. Knill

ABSTRACT. This is a proof of local existence of solutions of ordinary differential equations.

METRIC SPACES. Let X be a set on which a distance $d(x,y)$ between any two points x,y is defined. The function d must have the properties $d(y,x) = d(x,y) \geq 0$, $d(x,x) = 0$ and that $d(x,y) > 0$ for two different points x,y. Furthermore, one requires the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ to hold for all x,y,z. A pair (X,d) with these properties is called a **metric space**.

EXAMPLES. 1) The plane R^2 with the usual distance $d(x,y) = |x-y|$. An other metric is the Manhattan or taxi metric $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$.

2) The set $C([0,1])$ of all continuous functions $x(t)$ on the interval $[0,1]$ with the distance $d(x,y) = \max_t |x(t) - y(t)|$ is a metric space.

CONTRACTION. A map $\phi : X \to X$ is called a **contraction**, if there exists $\lambda < 1$ such that $d(\phi(x), \phi(y)) \leq \lambda \cdot d(x,y)$ for all $x,y \in X$. The map ϕ shrinks the distance of any two points by the contraction factor λ.

EXAMPLES. 1) The map $\phi(x) = \frac{1}{2}x + (1,0)$ is a contraction on R^2.
2) The map $\phi(x)(t) = \sin(t)x(t) + t$ is a contraction on $C([0,1])$ because $|\phi(x)(t) - \phi(y)(t)| = |\sin(t)| \cdot |x(t) - y(t)| \leq \sin(1) \cdot |x(t) - y(t)|$.

CAUCHY SEQUENCE. A **Cauchy sequence** in a metric space (X,d) is defined to be a sequence which has the property that for any $\epsilon > 0$, there exists n_0 such that $|x_n - x_m| \leq \epsilon$ for $n \geq n_0, m \geq n_0$.

COMPLETENESS. A metric space in which every Cauchy sequence converges to a limit is called **complete**.

EXAMPLES 1) $(R^n, d(x,y) = |x-y|)$ is complete. The rational numbers $(Q, d(x,y) = |x-y|)$ are not.

2) $C[0,1]$ is complete: given a Cauchy sequence x_n, then $x_n(t)$ is a Cauchy sequence in R for all t. Therefore $x_n(t)$ converges point-wise to a function $x(t)$. This function is continuous: take $\epsilon > 0$, then $|x(t) - x(s)| \leq |x(t) - x_n(t)| + |x_n(t) - y_n(s)| + |y_n(s) - y(s)|$ by the triangle inequality. If s is close to t, the second term is smaller than $\epsilon/3$. For large n, $|x(t) - x_n(t)| \leq \epsilon/3$ and $|y_n(s) - y(s)| \leq \epsilon/3$. So, $|x(t) - x(s)| \leq \epsilon$ if $|t-s|$ is small.

BANACH's FIXED POINT THEOREM. A contraction ϕ in a complete metric space (X,d) has exactly one fixed point in X.

PROOF.

(i) We first show by induction that

$$d(\phi^n(x), \phi^n(y)) \leq \lambda^n \cdot d(x,y)$$

for all n.

(ii) Using the triangle inequality and $\sum_k \lambda^k = (1-\lambda)^{-1}$, we get for all $x \in X$,

$$d(x, \phi^n x) \leq \sum_{k=0}^{n-1} d(\phi^k x, \phi^{k+1} x) \leq \sum_{k=0}^{n-1} \lambda^k d(x, \phi(x)) \leq \frac{1}{1-\lambda} \cdot d(x, \phi(x)) .$$

(iii) For all $x \in X$ the sequence $x_n = \phi^n(x)$ is Cauchy because by (i),(ii),

$$d(x_n, x_{n+k}) \leq \lambda^n \cdot d(x, x_k) \leq \lambda^n \cdot \frac{1}{1-\lambda} \cdot d(x, x_1) .$$

By completeness of X it has a limit $\tilde{x}$ which is a fixed point of ϕ.

(iv) There is only one fixed point. Assume, there were two fixed points $\tilde{x}, \tilde{y}$ of ϕ. Then

$$d(\tilde{x}, \tilde{y}) = d(\phi(\tilde{x}), \phi(\tilde{y})) \leq \lambda \cdot d(\tilde{x}, \tilde{y}) .$$

This is impossible unless $\tilde{x} = \tilde{y}$.

THE CAUCHY-PICARD EXISTENCE THEOREM. Assume $f : \mathbf{R}^n \to \mathbf{R}^n$ has a continuous derivative. For every initial condition x_0 there exists $\tau > 0$ such that on the time interval $[0,\tau)$ there exists exactly one solution of the initial value problem

$$\dot{x}(t) = f(x(t)), x(0) = x_0 .$$

PROOF.

(i)
Consider for every $\tau > 0$ and $r > 0$ the complete metric space

$$X = X_\tau(r) = \{x \in C[0,\tau] \mid \max_{0 \leq t \leq \tau} ||x(t) - x_0|| \leq r\}$$

with metric $d(x,y) = \max_{0 \leq t \leq \tau} ||x(t) - y(t)||$. With $c(t) = x_0$, we can write also $X = \{x \mid d(x,c) \leq r\}$.

Define a map ϕ on $C[0,\tau]$ by

$$\phi(y) : t \mapsto x_0 + \int_0^t f(y(s)) \, ds .$$

(ii) Define the constant

$$\lambda = \max\{\frac{||f(u) - f(v)||}{||u-v||} \mid ||u - x_0|| \leq 1, ||v - x_0|| \leq 1, u \neq v\} .$$

For every $x,y \in X_\tau(r)$ and $\tau \leq 1$, one has then

$$||f(x(s)) - f(y(s))|| \leq \lambda \cdot ||x(s) - y(s)|| \leq \lambda \cdot d(x,y)$$

for every $0 \leq s < \tau$. Therefore

$$d(\phi(x), \phi(y) = \max_{0<t<\tau} || \int_0^t f(x(s)) - f(y(s)) \, ds|| \leq \int_0^t ||f(x(s)) - f(y(s))|| \, ds \leq \lambda \tau d(x,y) .$$

We see that for small enough τ, the map ϕ is a contraction.

(iii) With $M = \max\{||f(x(t))|| \mid 0 \leq t \leq 1, \ d(x,c) \leq 1\}$, one has

$$||\phi(c) - c|| = || \int_0^t f(x_0(s)) \, ds|| \leq \int_0^t ||f(x_0(s))|| \, ds \leq M \cdot \tau .$$

If $\tau \leq 1$ is small enough, then $M \cdot \tau < (1-\lambda)r$. Using the triangle inequality, we obtain

$$d(\phi(x), c) \leq d(\phi(x), \phi(c)) + d(\phi(c), c) \leq \lambda d(x,c) + M\tau \leq \lambda r + (1-\lambda)r = r$$

proving that ϕ maps $X = \{d(x,c) \leq r\}$ into itself.

(iv) The fixed point ϕ in X obtained by Banach's fixed point theorem is a solution of the differential equation $\dot{x} = f(x)$ with initial value $x(0) = x_0$.

EXAMPLE WITH NO UNIQUE SOLUTION.

The differential equation $\frac{d}{dt}x = \sqrt{x}$ with $x(0) = 0$ has the solution $x(t) = Ct^2/4$ for any C. There are infinitely many solutions with the initial condition $x(0) = 0$. Note that the function $F(x)$ is not differentiable at $t = 0$.

EXAMPLE WITH NO GLOBAL SOLUTION.

The differential equation $\frac{d}{dt}x = x^2$ with initial condition $x(0) = 1$ has the solution $x(t) = 1/(1-t)$. At $t = 1$, the solution has escaped to infinity.

P.S. The photos show Stefan Banach (1892-1945), Emile Picard (1856-1941) and Augustin Cauchy (1789-1857).