DIDFFERENTIAL EQUATIONS IN TWO DIMENSIONS Math118, O. Knill

ABSTRACT. Differential equations in the plane do not show chaotic behavior. An interesting feature in two dimensions are limit cycles and their bifurcation. We look at some examples of such differential equations.

DIFFERENTIAL EQUATIONS. **Ordinary differential equations** are equations for an unknown function $x(t)$ in which which the derivatives with respect to one variable t appears. If derivatives with respect to several variables would occur, one would speak of partial differential equations. By introducing new variables for higher derivatives and possibly for time t, one can always bring it into the form

$$\frac{d}{dt} x(t) = f(x(t))$$

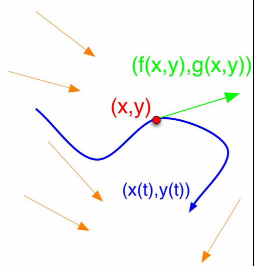
where $x(t)$ is a vector.

EXAMPLE. To write the second order inhomogeneous differential equation $\frac{d^2}{dt^2} x(t) + \frac{d}{dt} x(t) = \sin(t)$ in the above form, introduce $y(t) = \frac{d}{dt}$ and $z(t) = t$. Then $\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \sin(z(t)) - y(t) \\ 1 \end{bmatrix}$

DIFFERENTIAL EQUATIONS IN THE PLANE. A solution $\vec{x}(t)$ of a differential equation $\frac{d}{dt}\vec{x} = \vec{F}(\vec{x})$ is a vector quantity changing in time. The vector $F(\vec{x}(t))$ is the velocity vector. In two dimensions, we have

$$\begin{aligned} \dot{x}(t) &= f(x,y) \\ \dot{y}(t) &= g(x,y) \, . \end{aligned}$$

The **vector field** is obtained by attaching a vector $\vec{F}(x,y) = (f(x,y), g(x,y))$ to each point (x,y). Of special importance are **equilibrium points**. These are points, where the velocity is zero.

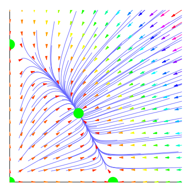


EXAMPLE. COMPETING SPECIES. A population of two species, where both compete for the same food can be modeled by the coupled logistic equations

$$\begin{aligned} \dot{x} &= \alpha x(1 - x/M) - \beta xy \\ \dot{y} &= \gamma y(1 - y/M) - \delta xy \, . \end{aligned}$$

A specific example is

$$\begin{aligned} \dot{x} &= 2x(1 - x/2) - xy \\ \dot{y} &= 3y(1 - y/3) - 2xy \end{aligned}$$

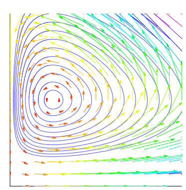


which has the equilibrium point $(1,1)$ because $(f(1,1), g(1,1)) = 0$. Additionally, one has the equilibrium points $(0,3), (2,0)$ and of course $(0,0)$.

EXAMPLE. PREDATOR-PREY. These systems of the form

$$\begin{aligned} \dot{x} &= \alpha x - \beta xy \\ \dot{y} &= -\gamma y + \delta xy \end{aligned}$$

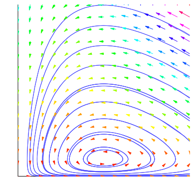
are also known under the name Volterra-Lodka systems. They can describe for example a shark-tuna population. The tuna population $x(t)$ becomes smaller with more sharks. The shark population $y(t)$ grows with more tuna. Historically, Volterra explained so the oscillation of fish populations in the Mediterrian sea. Here is a specific example:



$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \end{aligned}$$

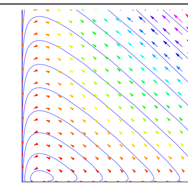
EXAMPLE. AIDS EPIDEMIC. The previous model can also model an epidemic as you can read in detail in Tom's lecture notes. In the interpretation of the epidemic, $x(t)$ is the size of the susceptible population, while $y(t)$ is the size of the infected population. A specific example modeling AIDS is

$$\begin{aligned} \dot{x} &= 0.2x - 0.1xy \\ \dot{y} &= -y + 0.1xy \end{aligned}$$

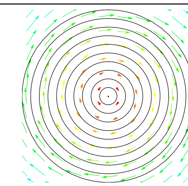


EXAMPLE. EBOLA EPIDEMIC. If the disease kills fast like in the case of ebola, we get a different picture

$$\begin{aligned} \dot{x} &= 0.2x - 0.5xy \\ \dot{y} &= -y + 0.5xy \end{aligned}$$



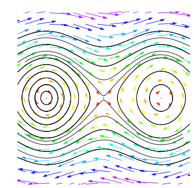
HARMONIC OSCILLATOR. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $\vec{x} = (x,y)$ be written as $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The direction field is always perpendicular to $\vec{x}$ so that by the product differentiation rule $d/dt \vec{x} \cdot \vec{x} = 2\vec{x}' \cdot \vec{x} = 0$ and $|\vec{x}|$ is constant. The solution curves are circles. In the homework, you look at a bit more general case. $\dot{x} = y, \dot{y} = -cx$, where c is a constant.



HAMILTONIAN SYSTEMS. If H is a function of two variables, we can look at the system

$$\begin{aligned} \dot{x} &= \partial_y H(x,y) \\ \dot{y} &= -\partial_x H(x,y) \end{aligned}$$

H is called the **energy** or Hamiltonian, x is called the position and y the momentum. Hamiltonian systems preserve energy $H(x,y)$: $\frac{d}{dt} H(x(t), y(t)) = \partial_x H(x,y)\dot{x} + \partial_y H(x,y)\dot{y} = \partial_x H(x,y)\partial_y H(x,y) - \partial_y H(x,y)\partial_x H(x,y) = 0$. The level curves of H are solution curves of the system. The time T maps are integrable. The illustration to the right shows the solution curves for the pendulum $H(x,y) = y^2/2 - \cos(x)$, where

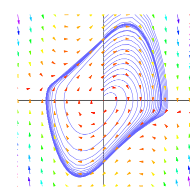


$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) \end{aligned}$$

Here x is the angle between the pendulum and y-axes, y is the angular velocity, $\sin(x)$ is the potential.

THE VAN DER POL EQUATION. $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ appears in electrical engineering, biology or biochemistry. It is an example of a **Lienhard system** differential equations of the form $\ddot{x} + \dot{x}F'(x) + G'(x) = 0$, where $F(x) = x^3/3 - x, g(x) = x$.

$$\begin{aligned} \dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -x \end{aligned}$$



Lienhard systems often have **limit cycles**, closed solution curves on which trajectories can be attracted to. Lienhard systems are useful for engineers, who need oscillators which are stable under random noise.