

2. Homework set

Math118, O.Knill

2.1 a) Realize the Henon map

$$T(x, y) = (y + 1 - ax^2, bx)$$

as a second order difference equation. A second order difference equation is a recursion of the form $x_{n+1} = F(x_n, x_{n-1})$.

b) An orbit x_0, x_1, x_2, \dots of a difference equation is called periodic, if there exists an integer n such that $x_{k+n} = x_k$ for all k .

Verify that periodic points of the difference equation define periodic points of T .

c) Find a periodic point of prime period 2 of the Henon map in the case $a = 1, b = 1$. The map is then

$$T(x, y) = (1 - x^2 + y, x)$$

. The notion **prime period 2** means that it should not be a fixed point.

d) (This is optional. Do it only if you have time and access to a CAS.) Can you find formulas for period 2 orbits for general a, b . You might need a computer algebra system. If you use a computer algebra system, find also all periodic orbits of the Henon map with prime period 3 and 4. The formulas can get messy.

2.2 a) Analyze the stability and nature of all the fixed points of the cubic Henon map $T(x, y) = (cx - x^3 - y, x)$ depending on the parameter c .

b) Find the **bifurcation points**, which are parameter values, where the stability of one of these fixed points changes.

2.3 Consider the map

$$T(x, y) = (2x + 3y, x + 2y)$$

on the torus.

a) Is T area preserving?

b) Verify that the fixed point $(0, 0)$ of T is hyperbolic. What are the stable and unstable manifolds of this fixed point?

c) Find the Lyapunov exponents of each orbit of T as well as the entropy, which is the average of the Lyapunov exponent.

d) Argue, why $T(x, y) = (2x + 3y + \epsilon \sin(x), x + 2y)$ has homoclinic points for small ϵ .

Remark. No formal proof is required in d). Just explain in words.

2.4 a) Compute the Lyapunov exponent of fixed points of the Henon map $T(x, y) = (1 - x^2 + y, x)$.

b) Compute the Lyapunov exponent of the periodic orbit you found in [2.1c].

c) What is the Lyapunov exponent of an initial point on the stable manifold to the periodic point you found in b)?

Remark. No computation is necessary in c).

2.5 a) Prove that the cat map $T(x, y) = (2x + y, x + y)$ on the torus is not integrable.

Remark. We will put a hint for this problem on the course website on Friday.

b) Show that the cat map $T(x, y) = (2x + y, x + y)$ defined on the plane is integrable.

Remark: It is possible to give an explicit integral for T but it is also possible just give arguments.