3. Homework set **Math118, O.Knill**

3.1 a) Is the flow generated by the differential equation

$$\frac{d}{dt}x = -2\sin(x+2y) - x\sin(xy), \frac{d}{dt}y = \sin(x+2y) + y\sin(xy)$$

area-preserving?

b) Find a function $H(x,y)$, such that the differential equation can be rewritten as

$$\frac{d}{dt}x = H_y(x,y), \frac{d}{dt}y = -H_x(x,y)$$

c) Is the system integrable?

3.2 a) Assume that $\dot{x} = F(x)$ is a differential equation defined in an annulus $A = \{1 < (x^2+y^2) < 4\ \}$ and assume that A is left invariant under the differential equation. Assume that $\text{div}(F)(x,y) < 0$ everywhere in the annulus. Prove that there can exist maximally one cycle in A.

b) Assume that $\dot{x} = F(x)$ is a differential equation defined in the disk $D = \{x^2+y^2 < 1\ \}$. Assume that this disk is left invariant under the differential equation. Assume that $\text{div}(F) < 0$ everywhere in the disk. Prove that there can not exist any limit cycle in D.

3.3 a) Verify **Dulacs criterion**: assume $\dot{x} = F(x)$ is a differential equation in a region D of the plane. If there exists a smooth function g, such that $\text{div}(gF(x))$ has no zeros in D, then there are no closed cycles in D.

b) Use Dulacs criterion to show that

$$\begin{aligned} \frac{d}{dt}x &= x(2-x-y) \\ \frac{d}{dt}y &= y(4x-x^2-3) \end{aligned}$$

has no closed cycles in the region $D = \{x > 0, y > 0\}$.

Hint. This is hard to guess: try $g(x,y) = 1/(xy)$.

3.4 a) The **clycolytic oscillator** is a model for the biochemical process **glycloysis**:

$$\begin{aligned} \frac{d}{dt}x &= -x + ay + x^2y \\ \frac{d}{dt}y &= b - ay - x^2y \end{aligned}$$

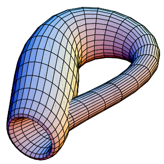
The system depends on two parameters $a > 0, b > 0$. The variable x is the concentration of ADP (adenosine diphosphate) and y is the concentration of F6P (fructose-6-phosphate). The parameter space is divided into two regions. One region, where the fixed point $(b, b/(a+b^2)$ is stable, the other, where the fixed point is unstable and where a stable limit cycle exists. Find the boundary between these two regions. When passsing this boundary, Hopf bifurcations occur.

b) Verify that the differential equation

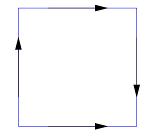
$$\begin{aligned} \frac{d}{dt}x &= y - (x^{11} - 100x) \\ \frac{d}{dt}y &= -x\ . \end{aligned}$$

has a unique limit cycle.

3.5 The **Klein bottle** is an example of a two-dimensional surface. It can not be realized without selfintersection in space. Explore whether the Poincare-Bendixon theorem holds on the Klein bottle or not.



Hint: you can build the Klein bottle as a square at which left and right are identified in the opposite orientation and top and bottom are identified with the same orientation. Start by gloing the top and bottom together. This gives a cylinder. Then, instead of glueing the cylinder together at the end (which produces a torus), glue them together in opposite direction.



3.6* (These are unsolved problems and therefore optional).
a) (**Dulac problem**) Verify that a differential equation

$$\begin{aligned} \frac{d}{dt}x &= p(x,y) \\ \frac{d}{dt}y &= q(x,y) \end{aligned}$$

with polynomials p and q of degree n has only finitely many limit cycles. Find a bound for their number $H(n)$.

b) (Special case of **Hilberts 16'th problem**) Show that a Lienard system with $g(x) = x$ and polynomial $F(x)$ of degree $2k+1$ has at most k limit cycles.