5. Homework set	Math118, O.Knill

5.1 Given a rectangle of length 1 and height $b > 1$. We play billiards in this table. For which angles θ does a trajectory (which does not hit a corner) get arbitrarily close to any point on the boundary of the table?

5.2 We have seen that for every period n with prime n, there is a periodic orbit of a biliard. We have done this by by maximizing the length functional $H(x_1, ..., x_n)$, which is the total length of the closed trajectory. We have assumed that the integer n has no non-trivial factor, because we did not want to have a periodic orbit of some smaller period.

a) Can you prove that the result actually holds for any n? There is a periodic orbit of minimal period n.

b) Prove that there are at least two periodic orbits of period 5 in the table $x^4 + y^4 \leq 1$.

5.3 Here is an application of the Kronecker dynamical system $x \to x + \alpha$. Consider the first digits of the powers

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ..$$

Can you determine how often each digit occurs in average? Which digit does occur more often, the digit 8 or the digit 9?

Hint. If a power 2^k starts with the digit 5 then $5 \cdot 10^m \leq 2^k < 6 \cdot 10^m$ for some m. Take logaritms to the base 10 of this equation and watch out for a Kronecker system. You are allowed to use **Weyls theorem** without proof, which assures that for irrational α, the frequency with which $[k\alpha]$ is in some interval $[a, b]$ is equal to $b - a$. We will prove that later.

5.4 a) A table is called convex, if the line segment connecting two arbitrary points in the table is inside the table. Verify that the billiard map can not be continuous on the annuous $(R/Z) \times [-1, 1]$, if the table is not convex.

b) Verify that the billiard in a half ellipse $x^2/a^2 + y^2/b^2 \leq 1, y \geq 0$ is integrable.

5.5 The string construction allows to construct a table, with a given caustic.

a) Draw a family of tables, which have a regular triangle as a caustic.

b) Draw a family of tables, which have a regular square as caustics.

c) Is there a convex billiard table which has two different caustics, where each caustic is a polygon?