STRICT ERGODICITY (* not treated in class) Math118, O. Knill

ABSTRACT. The irrational rotation on the circle is a minimal uniquely ergodic system. Other systems occuring in number theory have the same property.

ERGODICITY. A map T is ergodic if for every function $f(x) = \sum_{n \in Z} c_n e^{inx}$ with finite $\sum_n c_n^2$, the condition $f(T) = f$ implies $f =$ const.

THEOREM. For irrational α, the map $T(x) = x + \alpha$ is ergodic.

PROOF. Comparing Fourier coefficients of $f(T)$ and f gives $e^{in\alpha} c_n = c_n$ so that $c_n = 0$ unless $n = 0$.

UNIQUE ERGODICITY. A continuous transformation T on a compact topological space is called **uniquely ergodic** if there is only one invariant measure μ of T.

KRONECKER-WEYL THEOREM. The only measure which is invariant under an irrational rotation is the length measure dx.

PROOF. A measure μ is a linear map from the space of all continuous functions $C(X)$ to R given by $\mu(f) = \int f(x)\, d\mu(x)$. If μ is T invariant, then $\mu(f(T)) = \mu(f)$ and by linearity $\mu(\frac{1}{n}\sum_{k=1}^n f(T^k)) = \mu(f)$. Because for $f(x) = e^{ikx}$, we have

$$\frac{1}{n}\sum_{k=1}^n f(T^k) = \frac{1}{n}\sum_{k=1}^n e^{ij(x+k\alpha)} = \frac{e^{ijx}}{n}\frac{(1-e^{ijn\alpha})}{(1-e^{i\alpha})} \to 0$$

also for any $f = \sum_k e^{ikx}$ we have $\mu(f) = \mu(\frac{1}{n}\sum_{k=1}^n f(T^k)) \to c_0$ for $n \to \infty$ which implies $\mu(f) = c_0 = \int f(x)\, dx$.

MINIMALITY. A map T is called **minimal**, if every orbit of T is dense.

THEOREM. The irrational rotation on the circle is minimal.

PROOF. This follows in a constructive way from Chebychevs theorem. For every x and y and $\epsilon > 0$, there exists n such that $|x + n\alpha - y| < \epsilon$.

STRICT ERGODICITY. A map is called **strictly ergodic**, if it is both minimal and uniquely ergodic.

COROLLARY. The irrational rotation on the circle is strictly ergodic.

HIGHER DIMENSIONAL GENERALIZATION. The above statements go through word by word for a rotation $T(x) = x + \alpha$ with vectors $\alpha = (\alpha_1, ..., \alpha_d)$ for which $n \cdot \alpha = n_1\alpha_1 + ... + n_d\alpha_d) = 0$ implies $n = 0$. We call such vectors **irrational**. Functions of several variables have a Fourier expansion too: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\cdot x}$, where $n = (n_1, ..., n_d)$ runs over all lattice points in Z^d.

COROLLARY. The irrational translation on the torus $T^d = R^d/Z^d$ is strictly ergodic.

PROOF. We have shown both minimality as well as unique ergodicity.

THEOREM (FURSTENBERG) If α is irrational and $b_{ij} \in \mathbf{Z}, 1 \leq j < i \leq d$ real with $b_{i,i-1} \neq 0$. Then $T(x_1, \dots, x_d) = (x_1 + \alpha, x_2 + b_{21}x_1, ..., x_d + b_{d1}x_1 + ... + b_{d,d-1}x_{d-1})$ defines a uniquely ergodic system on $\mathbf{T}^d$.

It can be written as $\vec{x} \mapsto A\vec{x} + e_1\alpha$, where $A = \begin{bmatrix} 1 & 0 & . & 0 \\ b_{21} & 1 & . & 0 \\ b_{31} & b_{32} & . & . \\ . & . & . & . \\ b_{d1} & . & . & 1 \end{bmatrix}$.

PROOF. Fourier theory shows that T is ergodic: $f(T) = \sum_n c_n e^{in\cdot T(x)}$ with $n \cdot T(x) = (n_1, ..., n_d) \cdot (x_1 + \alpha, x_2 + b_{21}x_1, ..., x_d + b_{d1}x_1 + ... + b_{d,d-1}x_{d-1}) = n_1\alpha + An \cdot x$. Comparing Fourier coefficients gives $c_{An} = c_n e^{2\pi i n_1 \alpha}$ which implies $n = (n_1, 0, ..., 0)$ and therefore $c_n = c_n e^{2\pi i n_1 \alpha}$ which implies that $c_n = 0$ unless $n = 0$.

Unique ergodicity is shown with induction to d. We know it for $d = 1$, where the system is an irrational rotation. To prove the result in dimension d, write $T(\vec{x}, x_d) = (S(\vec{x}), x_d + A \cdot \vec{x})$. Note that S does not depend on x_d. By induction, S is uniquely ergodic on T^{d-1}. Given invariant measure μ for T, the projection of μ on $\mathbf{T}^{d-1}$ is S-invariant. and by induction assumption the volume measure $dx_1...dx_{d-1}$.

Because T commutes with $R(\vec{x}, y) = (\vec{x}, y + \beta)$, a T invariant measure must also be R_β invariant for every β. By Birkhoffs ergodic theorem, we know that μ almost all points $x = (\vec{x}, x_d)$ are **generic** in the sens that $\mu(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$. Assume $x = (\vec{x}, y)$ is generic. Then also $(\vec{x}, y + \beta)$ is generic.

A uniquely ergodic system on the torus which preserves the volume measure $dx_1...dx_n$ is automatically minimal: if there were an orbit x which were not dense, then its closure Y would be a T invariant set which is not the entire torus. This set would carry an other invariant measure.

ILLUSTRATION. Lets see this in the case $T(x, y) \to (x, x + y) \to (x + \alpha, x + y)$. When projecting onto the first coordinate, we have the uniquely ergodic map $x \to x + \alpha$. The key is that the map T commutes with $R(x, y) = (x, y + \beta)$:

$$T(R(x, y)) = T(x, y + \beta) = (x + \alpha, x + y + \beta), R(T(x, y)) = R(x + \alpha, x + y) = (x + \alpha, x + y + \beta) .$$

If (x_n, y_n) is an orbit, then the distribution of x_n on the first coordinate is the measure dx. Assume two different points $(x, y), (x, y + \beta)$ with irrational β produce measures $\mu(x, y), \mu(x, y + \beta)$ which must coincide.

APPLICATION: Let $p(x)$ be polynomial of degree n. Define $p_n(x) = p(x), p_{n-1} = p_n(x + 1) - p_n(x), p_{n-2} = p_{n-1}(x + 1) - p_{n-1}(x), .., p_0(x) = \alpha$. Each p_i is a polynomial of degree i. With

$$T_p = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 + \alpha \\ x_2 + x_1 \\ ... \\ x_n + x_{n-1} \end{bmatrix}$$

we have $T_p(p_1(n), p_2(n), ..., p_d(n)) = (p_1(n + 1), ..., p_d(n + 1))$.

CORROLLARY. If $p = a_n x^n + ... + a_1 x + a_0$ is a polynomial of degree n and assume a_n is irrational, then T_p is a uniquely ergodic transformation on the n dimensional torus which preserves the volume $\mu = dx_1...dx_n$.

QUESTION. Are polynomials the only functions f for which one can describe $f(n)$ mod 1 by a finite dimensional system?

EXAMPLES.
1) For $f(x) = \sqrt{x}$. What dynamical system does $\sqrt{n}$ mod 1 generate?
2) Does $f(x) = \exp(x)$ generate an infinite dimensional system?
3) If $f(x)$ is a k-periodic function, then $f(n)$ is periodic too For $f(x) = \sin(2\pi x\alpha)$ with irrational α, then $f(n)$ is an almost periodic sequence. The system on the torus $(x, y) \to (x + \alpha), \sin(2\pi x))$ allows to read of $f(n)$ in one coordinate.
4) For rational functions like $f(x) = x/(1 + x^2)$, the system has a fixed point which attracts all points.

OTHER STRICTLY ERGODIC SYSTEMS. Any factor of a strictly ergodic system is strictly ergodic. This applies to symbolic dynamics.

Doing symbolic dynamics with a strictly ergodic system produces strictly ergodic subshifts. Let $A_1, A_2, .., A_n$ be a partition of T^d into subsets, define $S(x)_n = k$ if $T^n(x) \in A_k$. This defines a subshift which is strictly ergodic.

EXAMPLES. Sturmean sequences $x_n = 1_A(x + n\alpha)$ and especially the Fibonacci sequence are uniquely ergodic subshifts. Applying cellular automata maps on such subshifts generates new subshifts which are strictly ergodic. CA maps preserve both minimality as well as unique ergodicity.